UNIT-3

Integral calculus

Jacobians

- help switch between domains $[\,dx\,dy = r\,dr\,d\theta\,]$
- If u and v are functions of x & y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v wrt x, y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Properties

(1) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$, then

$JJ' = 1$

(2) If u, v are functions of r, s and r, s are functions of x, y, then $\frac{\partial(u,v)}{\partial(x,y)}$ is

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$ → (chain rule for Jacobians)

(3) If u and v are functions of x, y, then the necessary and sufficient condition for the existence of a functional relationship of the form $f(u,v) = 0$ is that

$J = \frac{\partial(u,v)}{\partial(x,y)} = 0$ (i.e), u & v are called functionally dependent.

It is also true that if $J \neq 0$, then u and v are functionally independent.

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $J = \frac{\partial(x,y)}{\partial(r,\theta)}$ and $J' = \frac{\partial(r,\theta)}{\partial(x,y)}$ and P.T $JJ' = 1$

$J\left(\frac{x,y}{r,\theta}\right)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J = r\cos^2\theta + r\sin^2\theta = r \longrightarrow (1)$$

$$J' = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$x = r\cos\theta$ & $y = r\sin\theta$.

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$$

$$\frac{x}{y} = \cot\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$ $\frac{\partial r}{\partial y} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$

$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot y\left(\frac{-1}{x^2}\right)$ $\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$

$= \frac{-\cancel{x^2}}{x^2+y^2} \cdot \frac{y}{\cancel{x^2}}$ $= \frac{x^2}{y^2+x^2} \cdot \frac{1}{\cancel{x}}$

$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$ $\frac{\partial \theta}{\partial y} = \frac{x}{y^2+x^2}$

$$J' = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{y^2+x^2} \end{vmatrix}$$

$J' = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$ $= \frac{1}{\sqrt{x^2+y^2}}$

$J' = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

$J = r$ (eq ①)

$$\boxed{\therefore JJ' = \frac{r}{r} = 1}$$

2- If $x = e^{u}\sec V$ and $y = e^{u}\tan V$, find $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,V)}{\partial(x,y)}$ and show $JJ' = 1$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial V} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = e^{u}\sec V \qquad \frac{\partial y}{\partial u} = e^{u}\tan V$$

$$\frac{\partial x}{\partial v} = e^{u}\sec v\tan v \qquad \frac{\partial y}{\partial v} = e^{u}\sec^2 V$$

$$J = \begin{vmatrix} e^{u}\sec V & e^{u}\tan V \\ e^{u}\sec v\tan v & e^{u}\sec^2 v \end{vmatrix}$$

$$J = e^{2u}\sec^2 V - e^{2u}\sec v\tan^2 V = \boxed{e^{2u}\sec V}$$

$$\frac{x}{y} = \operatorname{cosec} V \Rightarrow \boxed{V = \sin^{-1}\left(\frac{y}{x}\right)}$$

$$x^2 - y^2 = e^{2u}(\sec^2 v - \tan^2 v)$$

$$\ln(x^2-y^2) = 2u \Rightarrow \boxed{u = \frac{\ln(x^2-y^2)}{2}}$$

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

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$J' = \begin{vmatrix} \end{vmatrix}$ $v = \sin^{-1}\left(\frac{y}{x}\right)$ $u = \frac{\ln(x^2-y^2)}{2}$

$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \cdot \frac{(-y)}{(x^2)}$ $\frac{\partial u}{\partial x} = \frac{\cancel{2}x}{\cancel{2}(x^2-y^2)}$

$\cancel{\frac{\partial v}{\partial y} = \frac{1}{\sqrt{1+\frac{y^2}{x^2}}}\left(\frac{1}{x}\right)}$ $\frac{\partial u}{\partial y} = \frac{-\cancel{2}y}{\cancel{2}(x^2-y^2)}$

$\frac{\partial v}{\partial x} = \frac{\cancel{x}}{\sqrt{x^2-y^2}} \cdot \frac{(-y)}{x^{\cancel{2}}}$ $\frac{\partial u}{\partial x} = \frac{x}{x^2-y^2}$

$\frac{\partial v}{\partial x} = \frac{-y}{x\sqrt{x^2-y^2}}$ $\frac{\partial u}{\partial y} = \frac{-y}{x^2-y^2}$

$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}} \cdot \frac{1}{x}$ $\frac{\partial v}{\partial y} = \frac{1}{\sqrt{x^2-y^2}}$

$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2-y^2}}$

$$J' = \begin{vmatrix} \frac{x}{x^2-y^2} & \frac{-y}{x\sqrt{x^2-y^2}} \\ \frac{-y}{x^2-y^2} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$J' = \frac{x^2}{x(x^2-y^2)^{3/2}} - \frac{y^2}{x(x^2-y^2)^{3/2}}$$

$$= \frac{x^2-y^2}{x(x^2-y^2)^{3/2}} = \frac{e^{2u}}{e^{u}\sec v\,(e^{3u})} = \frac{1}{e^{2u}\sec v}$$

$$JJ' = \frac{e^{2u}\sec v}{e^{2u}\sec v} = 1$$

3. If $u = x^2+y^2+z^2$, $v = xy+yz+zx$ and $w = x+y+z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = y+z; \quad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = x+z; \quad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = 2z; \quad \frac{\partial v}{\partial z} = x+y; \quad \frac{\partial w}{\partial z} = 1$$

$$J = \begin{vmatrix} 2x & y+z & 1 \\ 2y & x+z & 1 \\ 2z & x+y & 1 \end{vmatrix}$$

$\Rightarrow$ functions u, v, w are dependent

$$= 2\begin{vmatrix} x & y+z & 1 \\ y & x+z & 1 \\ z & x+y & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & 1 & 1 \\ y & 1 & 1 \\ z & 1 & 1 \end{vmatrix}(x+y+z) = 0$$

The functions u, v, w are dependent as $J = 0$.

$\therefore$ $\boxed{u = w^2 - 2v}$ Please be careful!!!

4. If $u = x+y+z$, $uv = y+z$, $uvw = z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$z = uvw$.
$y = uv - uvw$.
$x = u - uv + \cancel{uvw} - \cancel{uvw}$

$z = uvw$
$y = uv - uvw$
$x = u - uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & v-vw & vw \\ -u & u-uw & uw \\ 0 & -uv & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & v & vw \\ -u & u & uw \\ 0 & 0 & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & v & vw \\ 0 & u & uw \\ 0 & 0 & uv \end{vmatrix}$$

$$\boxed{J = u^2 v}$$

5. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Are the functions u & v functionally related? If so, find their relationship.

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(1)(1-xy)-(x+y)(-y)}{(1-xy)^2} & \frac{1}{1+x^2} \\ \frac{(1)(1-xy)-(x+y)(-x)}{(1-xy)^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(1-\cancel{xy})-(-\cancel{xy}-y^2)}{(1-xy)^2} & \frac{1}{1+x^2} \\ \frac{1-\cancel{xy}-(-x^2-\cancel{xy})}{(1-xy)^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1}{(1+x^2)} \\ \frac{1+x^2}{(1-xy)^2} & \frac{1}{(1+y^2)} \end{vmatrix}$$

$$J = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$ provided $xy \neq 1$

Hence, J' does not exist.

$\therefore$ u & v are functionally related.

~~u = tan~~

$v = \tan^{-1}x + \tan^{-1}y$

$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$

$\boxed{u = \tan v}$

No inverse relation

6. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

$r^2 = x^2 + y^2$

$$J = \underbrace{\frac{\partial(u,v)}{\partial(x,y)}}_{J_1} \cdot \underbrace{\frac{\partial(x,y)}{\partial(r,\theta)}}_{J_2}$$

$$J_1 = \begin{vmatrix} 2x & 2y \\ -2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4(x^2+y^2)$$

$$J_2 = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$J = 4(x^2+y^2)(r) = 4r^3$$

7. If $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$ and $w = \frac{4xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$J = \begin{vmatrix} -\frac{2yz}{x^2} & \frac{3z}{y} & \frac{4y}{z} \\ \frac{2z}{x} & -\frac{3xz}{y^2} & \frac{4x}{z} \\ \frac{2y}{x} & \frac{3x}{y} & -\frac{4xy}{z^2} \end{vmatrix} = 24 \begin{vmatrix} \frac{yz}{x^2} & \frac{z}{y} & \frac{y}{z} \\ \frac{z}{x} & -\frac{xz}{y^2} & \frac{x}{z} \\ \frac{y}{x} & \frac{x}{y} & -\frac{xy}{z^2} \end{vmatrix}$$

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$$= \frac{24}{xyz} \begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{xz}{y} & +x \\ y & x & -\frac{xy}{z} \end{vmatrix}$$

$$= \frac{24}{(xyz)^2} \begin{vmatrix} -yz & zx & yx \\ zy & -xz & +xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{24}{(xyz)^2} \begin{vmatrix} 0 & 0 & 2yx \\ zy & -xz & +xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{24}{(xyz)^2} (2yx)\left(z^2yx + z^2y\cancel{z}\right)$$

$$= \frac{24}{\cancel{x^2y^2z^2}} \cdot 2y\cancel{x} \cdot 2\cancel{z}^2y\cancel{x})$$

$$\boxed{J = 96}$$

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$\cancel{u^2 = x^2 - y^2 \quad v = 2xy}$

$u = \frac{2yz}{x}$ $v = \frac{3xz}{y}$ $\omega = \frac{4xy}{z}$

$\cancel{x = \frac{2yz}{u}}$ $\cancel{y = \frac{3xz}{v}}$

$\cancel{y = \frac{3\left(\frac{2yz}{u}\right)z}{v}}$

$x^2u = 2xyz$ $y^2v = 3xyz$ $z^2\omega = 4xyz$

$\frac{x^2u}{2} = \frac{y^2v}{3} = \frac{z^2\omega}{4}$

$x = \frac{2u}{u}\sqrt{\frac{v\omega}{48}}$

$x = \frac{2yz}{u}$

$\boxed{x = \sqrt{\frac{v\omega}{12}}}$

$v = \frac{3z}{\cancel{y}}\left(\frac{2\cancel{y}z}{u}\right) = \frac{6z^2}{u} \Rightarrow \boxed{\sqrt{\frac{uv}{6}} = z}$

$z = \frac{4xy}{\omega}$

$\boxed{\sqrt{\frac{u\omega}{8}} = y}$

$\sqrt{\frac{uv}{6}} = \frac{4}{\omega}\left(\frac{2yz}{u}\right)(y)$

$\cancel{\sqrt{\frac{uv}{6}}} = \frac{4}{\omega}\frac{2}{u}\cancel{\sqrt{\frac{uv}{6}}}\, y^2$

$\cancel{4\omega}$ $\cancel{4u}$

Double Integrals

Let $f(x,y)$ be a continuous function of two independent variables x and y defined at every point of the region R of the xy plane.
Divide the region into n elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$.

Let (x_r, y_r) be any point in the area δA_r.
Consider the sum $\sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$

If $\lim_{n \to \infty} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$ exists finitely

and uniquely, it is called the double integral of $f(x,y)$ over the region R, written as

$$\iint_R f(x,y)\,dA$$

$$= \iint_R f(x,y)\,dx\,dy$$

Types of Double Integrals

TYPE 1: When all the four limits of integration are constants.

Case I: When $f(x,y)$ breaks into 2 factors, one for each variable.

Procedure:
In this case, the double integral can be broken into a product of 2 single integrals. While doing this, the inner limits go with the inner variable and the outer limits go with the outer variable

Illustration (boundaries form a square)

Q: Evaluate $\displaystyle\int_3^4 \int_1^2 \frac{dy\,dx}{xy^2}$

A: $\displaystyle I = \left(\int_1^2 \frac{dy}{y^2}\right)\left(\int_3^4 \frac{dx}{x}\right)$

$\displaystyle = \left[\frac{-1}{y}\right]_1^2 \cdot \left[\ln x\right]_3^4$

$\displaystyle = \left(1 - \tfrac{1}{2}\right)\left(\ln \frac{4}{3}\right) = \left(\frac{1}{2} \ln \frac{4}{3}\right)$

Case II: When $f(x,y)$ does not break into a product of 2 functions, one for each variable

Procedure:
In this case, we perform inner integration with respect to the inner limits, treating the outer variable as a constant. The resulting function is then integrated with respect to the outer limits.

Illustration:
Q: Evaluate $\int_3^4 \int_1^2 \frac{dy\,dx}{(x+y)^2}$

A.
$$I = \int_3^4 \int_1^2 \frac{dy}{(x+y)^2}\,dx$$

treat x as a constant

$$= \int_3^4 \left[\frac{-1}{x+y}\right]_1^2 dx$$

$$= \int_3^4 \left(\frac{-1}{x+2} + \frac{1}{x+1}\right) dx$$

$$= \Big[-\ln(x+2)\Big]_3^4 + \Big[\ln(x+1)\Big]_3^4$$

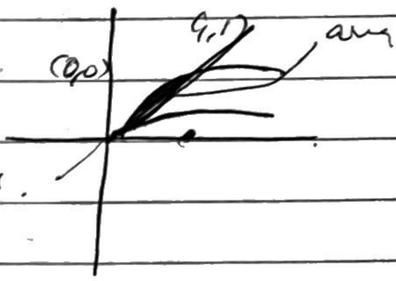
$$= \ln(5) - \ln(6) + \ln(5) - \ln(4)$$

$$I = \ln\left(\frac{25}{24}\right)$$

TYPE 2: When the outer limits are constants and the inner limits are functions.

Procedure:

In this case, the inner limits will be functions of the outer variable. We, therefore, perform the inner integration wrt the inner limits, treating the outer variable as a constant. The resulting function can then be integrated wrt the outer limits.



Illustration

Q: Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$.

$$I = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^2\sqrt{x} - x^3 + \frac{(\sqrt{x})^3}{3} - \frac{x^3}{3}\right) dx$$

$$= \int_0^1 \left(x^{5/2} - \frac{4}{3}x^3 + \frac{x^{3/2}}{3}\right) dx$$

$$= \left[\frac{x^{7/2}}{7/2} - \frac{\cancel{4}}{3}\frac{x^4}{\cancel{4}} + \frac{x^{5/2}}{3\times 5/2}\right]_0^1$$

$$= \left[\frac{2}{7}x^{7/2} - \frac{x^4}{3} + \frac{2}{15}x^{5/2}\right]_0^1 = \frac{2}{7} - \frac{1}{3} + \frac{2}{15}$$

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$$= \frac{2}{7} - \frac{1}{3} + \frac{2}{15} = \frac{2}{7} - \frac{5}{15} + \frac{2}{15}$$

$$= \frac{2}{7} - \frac{3}{15} = \frac{2}{7} - \frac{1}{5} = \frac{10-7}{35} = \frac{3}{35}$$

$$\boxed{I = \frac{3}{35}}$$

TYPE 3: when the limits of integration are not given but only the region is indicated

Procedure:
In this case, one of the variables can be chosen as the outer variable and its minimum and maximum values in the region can be found.
The inner variable can then be written in terms of the outer variable using the given equations.

Illustration

Q: $\iint_R xy(x+y)\,dx\,dy$ where R is the region bounded by the curves $y = x^2$ and $y = x$

Solution: (need to draw)

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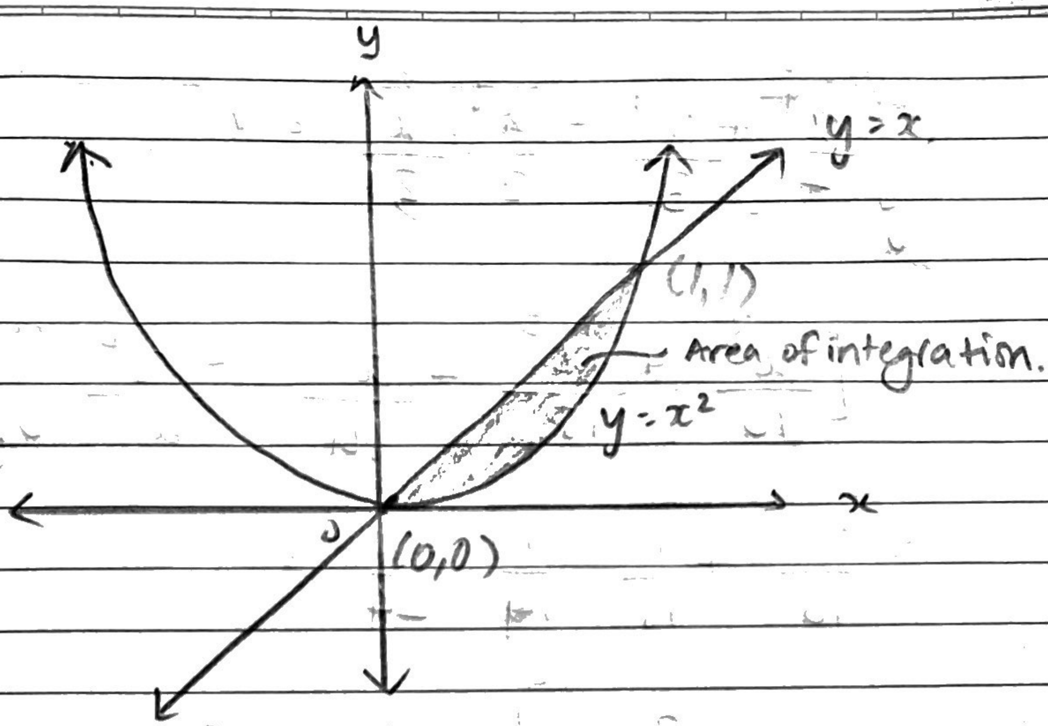

Equating $y = x$ and $y = x^2$

$x = x^2 \Rightarrow x^2 - x = 0 = x(x-1) = 0$

$\Rightarrow x = 1$ or $x = 0$.

$y = 1$ $\quad$ $y = 0$.

$(x, y) = (0,0)$ and $(1,1)$

Method 1:

Let x be the outer variable. x varies from 0 to 1

x^2 is below x.

$\therefore y$ from x^2 to x

$$I = \int_0^1 \int_{x^2}^{x} xy(x+y)\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} x^2y + xy^2\, dy\, dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx$$

$$\int_0^1 \frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \, dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{30} + \frac{2}{30} - \frac{1}{14} - \frac{1}{24} = \frac{5}{30} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{25} = \frac{-3}{42} + \frac{7}{42} - \frac{1}{29}$$

$$= \frac{4}{42} - \frac{1}{24} = \frac{2}{21} - \frac{1}{24} = \cancel{\frac{50-21}{21\times 25}}$$

$$\cancel{= \frac{9}{21\times 25} = \frac{3}{7\times 25}}$$

$$= \frac{2}{21} - \frac{1}{24} = \frac{48-21}{21\times 24} = \frac{27}{21\times 24} = \frac{9^3}{7\times 24_8}$$

$$= \frac{3}{56}$$

Method 2:

let y be the outer variable

$\cancel{y}$ $\quad y = x \quad y = x^2$

$x = y \quad x = \sqrt{y}$

y varies from 0 to 1

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$$I = \int_0^1 \int_y^{\sqrt{y}} x^2y + xy^2 \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2} \right]_y^{\sqrt{y}} dy$$

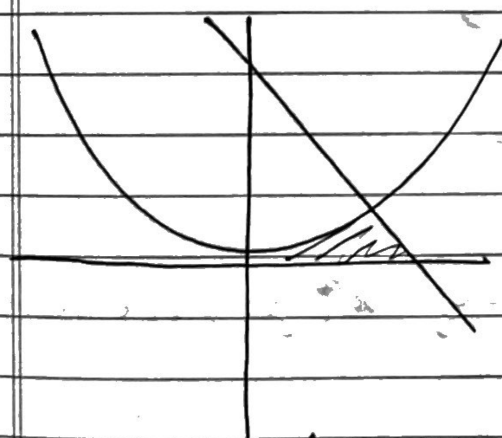
$$= \int_0^1 \frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{y^4}{3} - \frac{y^4}{2} \, dy$$

$$= \left[\frac{2y^{7/2}}{7\times 3} + \frac{y^4}{8} - \frac{y^5}{15} - \frac{y^5}{10} \right]_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{15} - \frac{1}{10} = \frac{2}{21} + \frac{1}{8} - \frac{1}{6}$$

$$= \frac{2}{21} + \frac{3}{24} - \frac{4}{24} = \frac{2}{21} - \frac{1}{24} = \frac{48-21}{21\times 24}$$

$$= \frac{\cancel{9}\ \cancel{27}\ 3}{7\ \cancel{21}\times\cancel{24}\ 8} = \frac{3}{56}$$



$\iint f(x)\, dx\, dy$

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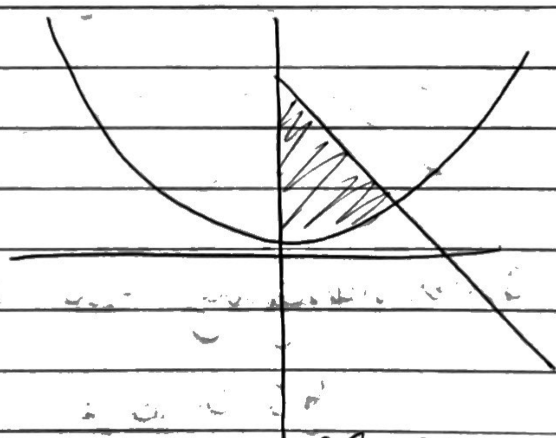
(exit boundary for inner function unchanged)



$\iint f(x)\, dy\, dx$

faster

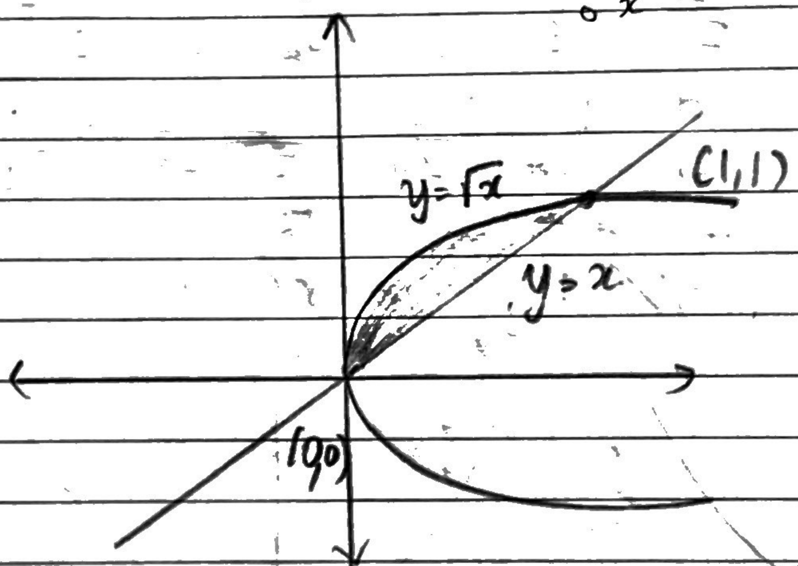
TYPE 4: Change of order of integration

Procedure:
In order to change the order of integration, we first identify the region with the help of the given limits. We then interchange the roles of the variables and find the new limits.

Illustration.

Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing the order of integration.

$$I = \int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$$



Interchanging the roles of x & y, we have

y: 0 to 1

x: y^2 to y

$$\therefore I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy$$

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$$I = \int_{y=0}^{1} \left[\frac{x^2 y}{2}\right]_{y^2}^{y} dy = \int_0^1 \left\{\frac{y^3}{2} - \frac{y^5}{2}\right\} dy$$

$$= \left[\frac{y^4}{8} - \frac{y^6}{12}\right]_0^1 = \frac{1}{8} - \frac{1}{12} = \frac{3-2}{24} = \frac{1}{24}$$

$$\boxed{I = 1/24}$$

TYPE 5: Change of Variables (coordinate system)

Procedure:

In order to change the variables x, y to the new variables u, v, we write $x = x(u, v)$ and $y = y(u, v)$, and find $J\frac{(x, y)}{(u, v)}$

Then, $dx\,dy = J\,du\,dv$
↳ transformation factor

$$\therefore \iint_{R_{xy}} f(x, y)\,dx\,dy = \iint_{R_{uv}} g(u, v)\,J\,du\,dv.$$

Illustration:

Evaluate $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

Note $\displaystyle\int_0^\infty e^{-x^2} dx = \Gamma(n)$ Gamma function

with $x = y^2$

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$$\int_0^\infty e^{-x^2}\,dx \longrightarrow$$ Gamma function.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx.$$

if $n = 1/2$

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2}\,dx$$

let $y^2 = x$

$2y\,dy = dx$

$dy = \frac{dx}{2\sqrt{x}}$

$$= 2\int_0^\infty e^{-y^2}\,dy$$

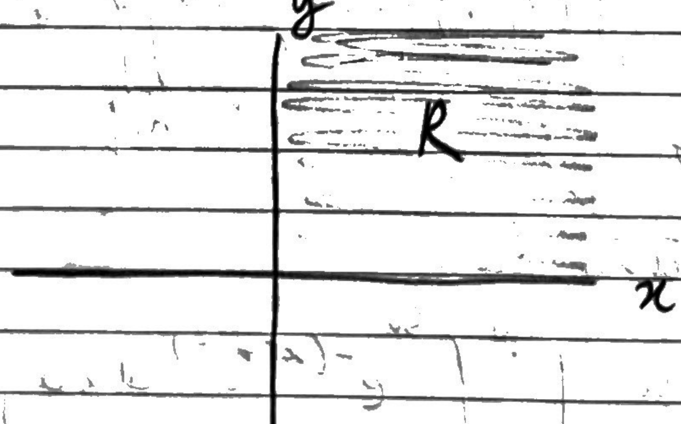
$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

<u>Solution:</u> using $x = r\cos\theta$,
$y = r\sin\theta$.

we get. $x^2+y^2 = r^2$
and $dx\,dy = J\,dr\,d\theta$.

where $J = r$ (refer prob 1, Jacobians)

$x : 0$ to ∞] first
$y : 0$ to ∞] quadrant



$r : 0$ to ∞
$\theta : 0$ to $\pi/2$.

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta.$$

$$I = \int_0^{\pi/2} \left[\frac{-e^{-r^2}}{2} \right]_0^{\infty} d\theta = \int_0^{\pi/2} \left[\frac{-1}{2e^{r^2}} \right]_0^{\infty} d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} \left[\frac{1}{e^{r^2}} \right]_0^{\infty} d\theta = \frac{-1}{2} \int_0^{\pi/2} 0 - 1 \, d\theta.$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}.$$

$$\therefore \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}$$

$$\Rightarrow \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right) = \frac{\pi}{4}.$$

y can be replaced with x.

$$\left(\int_0^{\infty} e^{-x^2} dx \right)^2 = \frac{\pi}{4}$$

$$\boxed{\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}} = \frac{1}{2} \Gamma(1/2)$$

30.09.19

8. Evaluate $\int_1^2 \int_3^4 (xy + e^y)\,dy\,dx$

$$I = \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx$$

$$I = \int_1^2 \left(\frac{16x}{2} + e^4 - \frac{9x}{2} - e^3\right) dx$$

$$= \int_1^2 \left(\cancel{8x - 24x}\ \frac{7x}{2} + e^4 - e^3\right) dx$$

$$= \int_1^2 -\frac{7x}{2} + e^4 - e^3\, dx$$

$$= \left[\frac{-7x^2}{4} + e^4 x - e^3 x\right]_1^2$$

$$\cancel{= \left[-8x^2 + x(e^4 - e^3)\right]_1^2 = \frac{-32 + (e^4 - e^3)8}{8}}$$

$$\cancel{= -24 + e^4 - e^3}$$

$$= \frac{-7}{4}(4-1) + e^4 - e^3 = \boxed{\frac{-21}{4} + e^4 - e^3}$$

9. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$\cancel{I_0} = \int_0^1$ let $1+x^2 = a^2$

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{a^2+y^2}\, dy\, dx = \int_0^1 \left[\frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left[\frac{1}{a}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$\cancel{= \int_0^1 \frac{1}{a}\left(\tan^{-1} 1 - \tan^{-1} 0\right) dx = \frac{1}{a}\int_0^1 \frac{\pi}{4}\, dx}$$

$$I = \frac{\pi}{4} \times 1$$

$$I = \int_0^1 \frac{\pi}{4} \times \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4}\left[\ln\left(x+\sqrt{1+x^2}\right)\right]_0^1 = \frac{\pi}{4}\left(\ln(1+\sqrt{2}) - \ln 1\right)$$

$$= \frac{\pi}{4}\ln(1+\sqrt{2})$$

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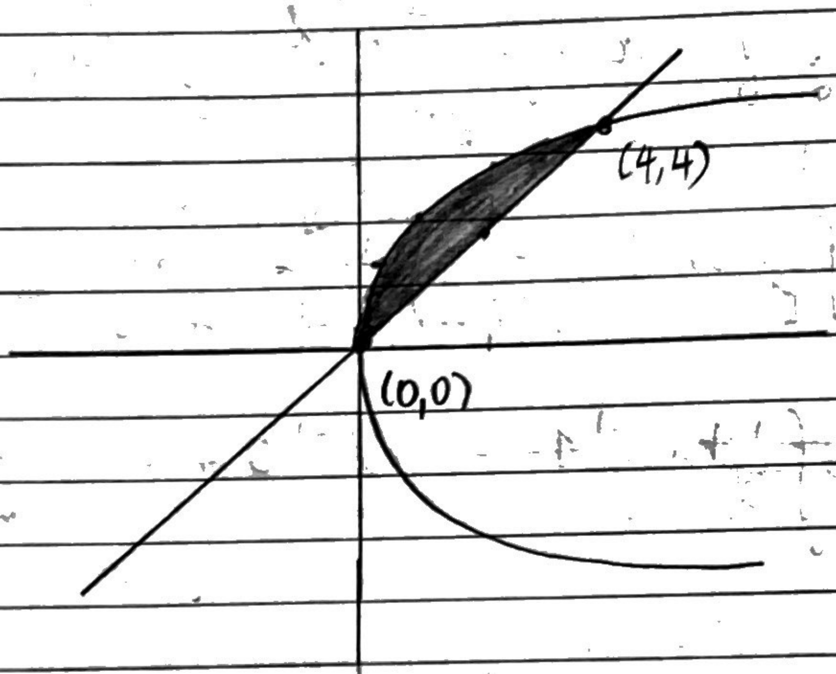
10. Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region bounded by $y=x$ and $y^2=4x$



$$\frac{y^2}{4} = y \Rightarrow y = 0 \text{ or } y = 4$$

$$y = 4 \Rightarrow x = 4$$

$$\int_0^4 \int_{y^2/4}^{y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{y^2/4}^{y} dy$$

$$= \int_0^4 \left(\frac{y^3}{3} + y^3 - \frac{y^6}{64\times 3} - \frac{y^4}{4} \right) dy$$

$$= \left[\frac{y^4}{12} + \frac{y^4}{4} - \frac{y^7}{64\times 7\times 3} - \frac{y^5}{4\times 5} \right]_0^4$$

$$= \frac{4^{\cancel{4}\,3}}{\cancel{4}\times 3} + \frac{4^{\cancel{4}\,3}}{\cancel{4}} - \frac{4^{\cancel{7}\,6}}{\cancel{4}\times 21\times 16} - \frac{4^{\cancel{5}\,4}}{\cancel{4}\times 5}$$

$$= \frac{64}{3} + 64 - \frac{4^6}{21\times 16} - \frac{4^4}{5}$$

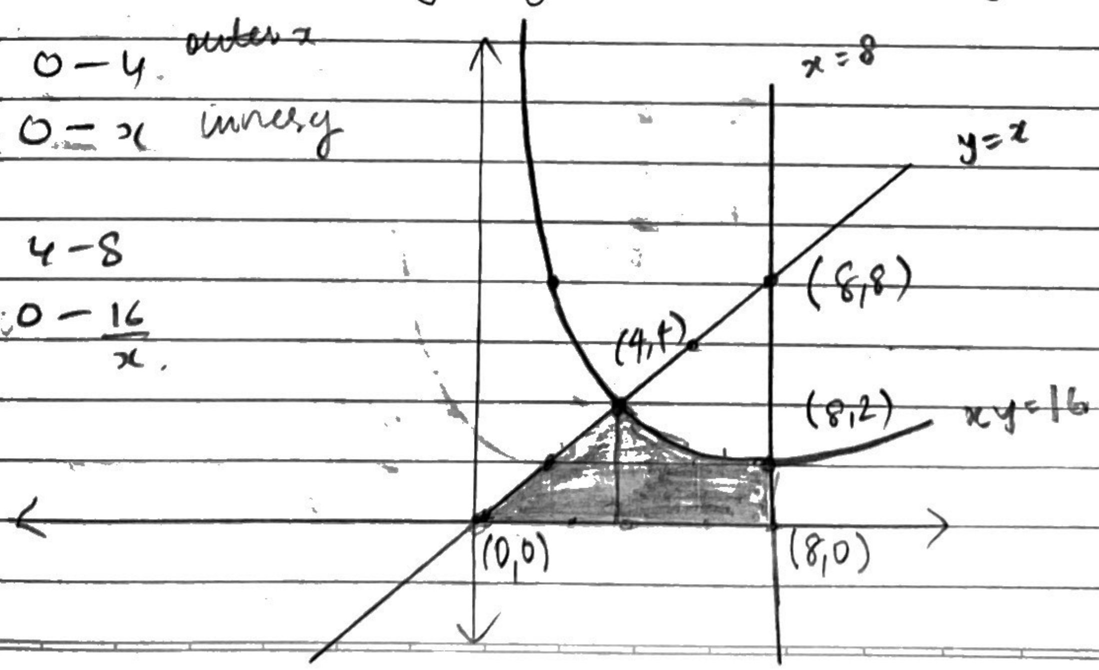
$$= \frac{256}{3} - \frac{4096}{21\times 16} - \frac{256}{5} = \cancel{\frac{4736}{35}}$$

$$= \frac{256}{3} - \frac{256}{21} - \frac{256}{5} = \cancel{\frac{1664}{35}} \quad \frac{768}{35}$$

11. Evaluate $\iint_R x^2 \, dx\, dy$ where R is the region in the first quadrant bounded by $x = y$, $y = 0$, $x = 8$ and $xy = 16$

$x : 0 - 4$ outer x
$y : 0 - x$ inner y

$x : 4 - 8$
$y : 0 - \frac{16}{x}$



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$$I = \int_0^4 \int_0^x x^2 \, dy \, dx + \int_4^8 \int_0^{16/x} x^2 \, dy \, dx$$

$$= \int_0^4 \left[x^2 y\right]_0^x dx + \int_4^8 \left[x^2 y\right]_0^{16/x} dx$$

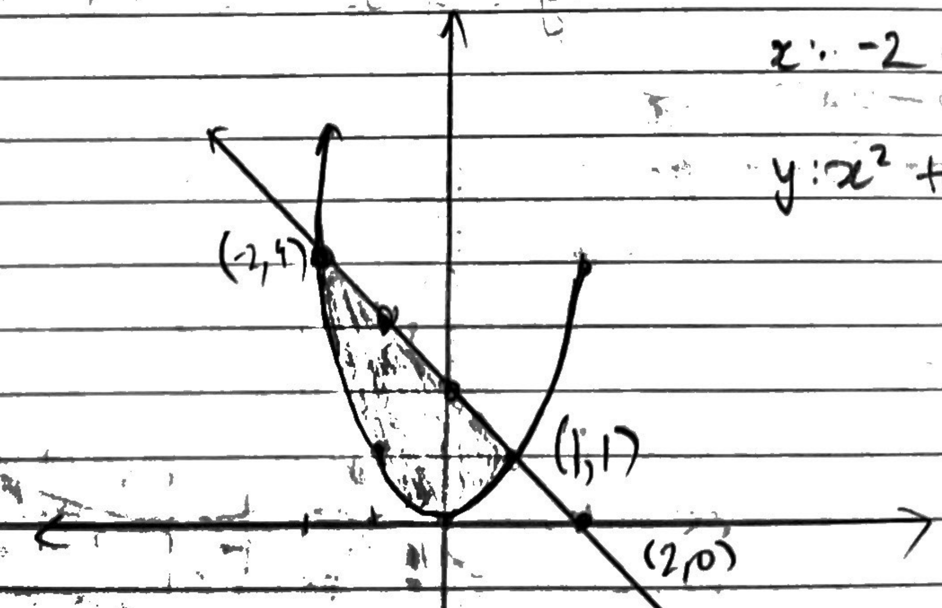
$$= \int_0^4 x^3 \, dx + \int_4^8 16x \, dx$$

$$= \left[\frac{x^4}{4}\right]_0^4 + \left[8x^2\right]_4^8 =$$

$$= 64 + 8(8^2 - 4^2) = 64 + 8 \times 16(3)$$

$$= 64 + 384 = 448$$

12. Evaluate $\iint_R y \, dx\, dy$ over the region bounded by $y = x^2$ and $x + y = 2$



x : -2 to 1

y : x^2 to $2-x$

$$I = \int_{-2}^{1}\int_{x^2}^{2-x} y \, dy\, dx = \int_{-2}^{1}\left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1}\left(\frac{(2-x)^2}{2} - \frac{x^4}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{1} (2-x)^2 - x^4 \, dx = \frac{1}{2}\left[\frac{(2-x)^3}{-3} - \frac{x^5}{5}\right]_{-2}^{1}$$

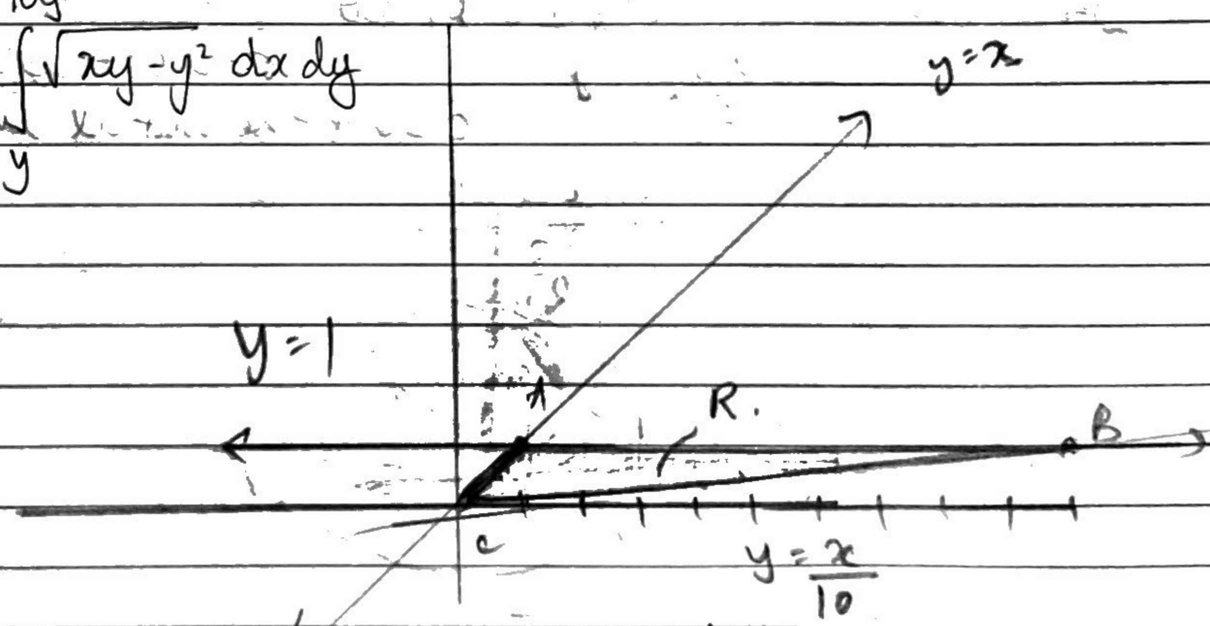
$$= \frac{1}{2}\left(\frac{(2-1)^3}{-3} - \frac{1}{5} - \frac{(2+2)^3}{-3} + \frac{(-2)^5}{5}\right)$$

$$= \frac{1}{2}\left(\frac{-1}{3} - \frac{1}{5} + \frac{4^3}{3} - \frac{32}{5}\right)$$

$$= \frac{1}{2}\left(\frac{-33}{5} + \frac{63}{3}\right) = \frac{36}{5}$$

13. Evaluate $\iint_R \sqrt{xy-y^2}\, dy\, dx$ where R is the triangle with vertices (0,0), (10,1), (1,1)

$$I = \int_0^1\int_y^{10y} \sqrt{xy - y^2}\, dx\, dy$$



x: y to 10y
y: 0 to 1

$$I = \int_0^1 \int_y^{10y} \sqrt{xy-y^2}\, dx\, dy$$

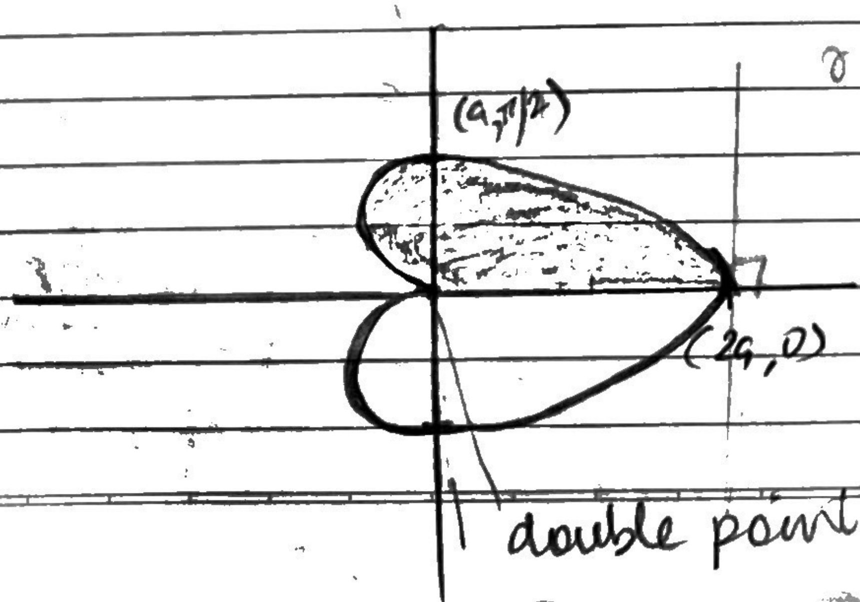
$$= \int_0^1 \left[\frac{(xy-y^2)^{3/2}}{y \times 3/2} \right]_y^{10y} dy$$

$$= \int_0^1 \frac{(10y^2-y^2)^{3/2}}{y \times 3/2} - \cancel{\frac{(y^2-y^2)^{3/2}}{y \times 3/2}}\, dy$$

$$= \int_0^1 \frac{\cancel{27}^{9}\, y^{\cancel{3}^{2}} \times 2}{\cancel{3} \times \cancel{y}}\, dy = \int_0^1 9y^2 \times 2\, dy$$

$$= 18 \int_0^1 y^2\, dy = \left[\frac{y^3}{3}\right]_0^1 \times 18 = \boxed{6}$$

14. Evaluate $\iint_R r^2 \sin\theta \, dr\, d\theta$ over the cardiod $r = a(1+\cos\theta)$ above the initial line



$r = a(1+\sin\theta)$
$\pm \sin$
or $\cos$

θ : ~~r 8 2a...~~

θ : 0 to π

Limits for r : any direction
0 to $a(1+\cos\theta)$

$$I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr \, d\theta.$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} \frac{\sin\theta}{3} \left(a^3(1+\cos\theta)^3\right) d\theta = \frac{a^3}{3}\int_0^{\pi} \cancel{\sin\theta(1+\cos^3\theta+3\cos^2\theta+3\cos\theta)\,d\theta}$$

$$= \cancel{\frac{a^3}{3}\int_0^{\pi}} \cancel{\sin\theta + \sin\theta\cos^3\theta + 3\sin\theta\cos^2\theta} + 3\sin\theta\cos\theta \, d\theta$$

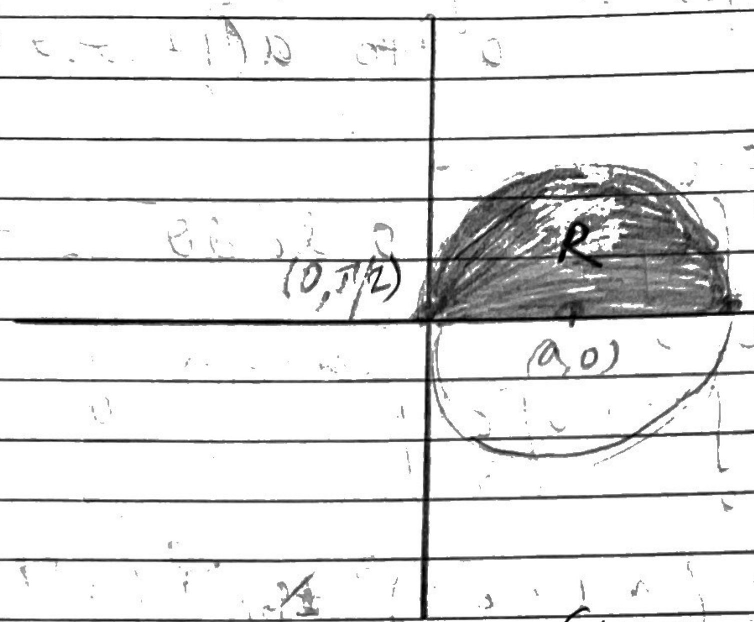
$=$ let $1+\cos\theta = t$ $\qquad \theta = 0 ; t = 2$

$-\sin\theta \, d\theta = dt \qquad \theta = \pi ; t = 0.$

$$\int_2^0 \frac{-a^3}{3} t^3 \, dt = \frac{-a^3}{3}\left[\frac{t^4}{4}\right]_2^0 = \frac{a^3}{3} \times 4$$

$$\boxed{I = \frac{4a^3}{3}}$$

15. Evaluate $\iint_R r^2 \sin\theta \, dr \, d\theta$ where R is the region bounded by $r = 2a\cos\theta$ above the initial line.



$r = 2a\cos\theta$

$r^2 = 2ar\cos\theta$

$x^2 + y^2 = 2ax$

$(x-a)^2 + y^2 = a^2$

$\theta : 0 \longrightarrow \pi/2$ (NOT $0 - \pi$)

$$I = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3} \right]_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta \; 8a^3 \cos^3\theta}{3} \, d\theta$$

$t = \cos\theta$

$dt = -\sin\theta \, d\theta$

$\theta = 0 ; t = 1$

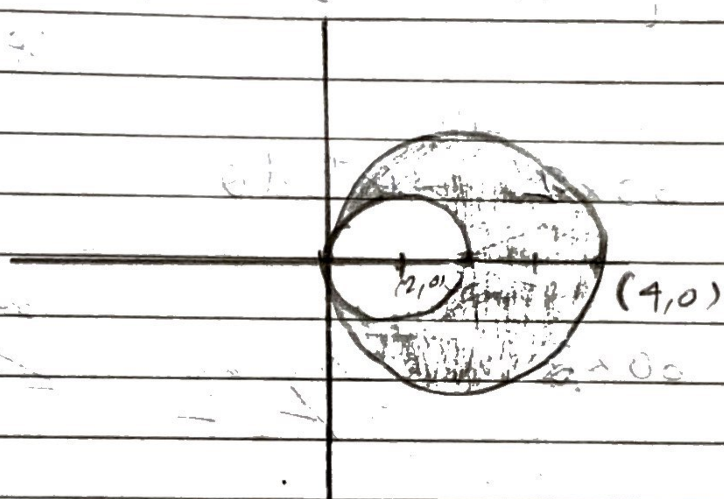
$\theta = \pi/2 ; t = 0$

$$= \int_1^0 -\frac{8a^3 t^3}{3} \, dt$$

$$= \int_0^1 \frac{8a^3t^3}{3}\,dt = \frac{8a^3}{3}\left[\frac{t^4}{4}\right]_0^1 = \cancel{8}$$

$$= \boxed{\frac{2}{3}a^3}$$

16. Evaluate $\iint_R r^3\,dr\,d\theta$ over the area bounded between $r = 2\cos\theta$ and $r = 4\cos\theta$



$$\cancel{I = \int_{-\pi/2}^{\pi/2}}$$

$$I = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2}(4^4 - 2^4)\cos^4\theta\,d\theta$$

$$= 60\int_{-\pi/3}^{2\pi/2}\cos^4\theta\,d\theta = \cancel{60\int_0^{2\pi}\cos}$$

Reduction formula

has to be +ve

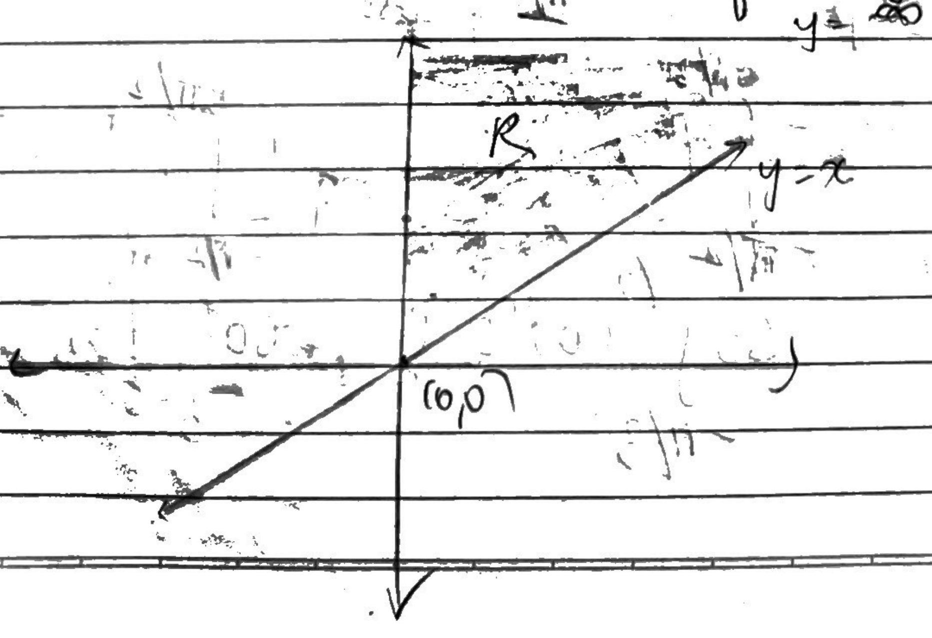
$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \int_0^{\pi/2} \cos^n \theta \, d\theta$$

$$= \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even.} \\[2ex] \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{4}{5} \cdot \dfrac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

$$= 60 \times 2 \times \int_0^{\pi/2} \cos^4 \theta \, d\theta$$

$$= 60 \times \cancel{2} \times \frac{3}{4} \times \frac{1}{\cancel{2}} \times \frac{\pi}{2} = \frac{\overset{15}{\cancel{\overset{30}{\cancel{60}}}} \times 3\pi}{8\,\cancel{4}\,2} = \boxed{\frac{45\pi}{2}}$$

17. Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y} \, dy \, dx$ by changing the order of integration



Changing the order of integration

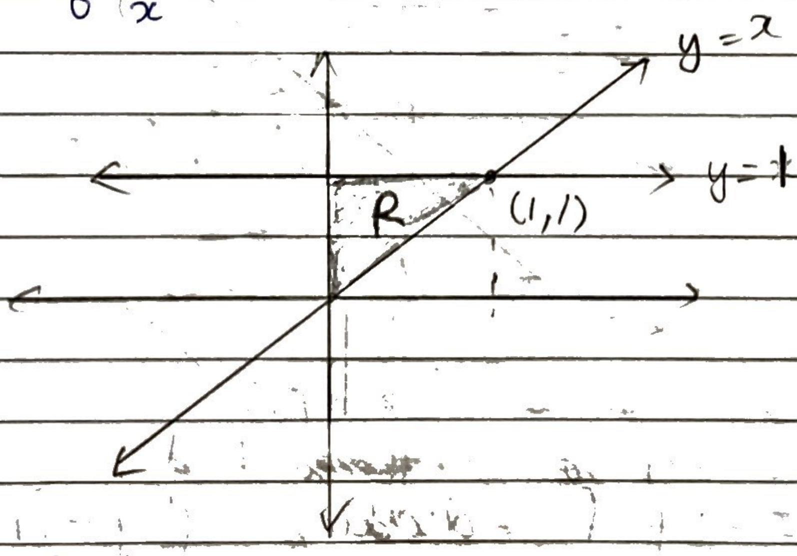
$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$\int_0^\infty \left[\frac{e^{-y} x}{y}\right]_0^y dy = \int_0^\infty e^{-y} dy$$

$$= \left[-e^{-y}\right]_0^\infty = -e^{-\infty} + e^{0}$$

$$= -\frac{1}{e^\infty} + 1 = 1$$

18. Evaluate $\int_0^1 \int_x^1 \sin y^2 \, dy\, dx$. by changing order.



Changing order,

$$I = \int_0^1 \int_0^y \sin y^2 \, dx\, dy = \int_0^1 \left(\left[\sin y^2 \; x\right]_0^y \right) dy$$

$$I = \frac{1}{2}\int_0^1 2y \sin y^2 \, dy = \cancel{\tfrac{1}{2}\int}$$

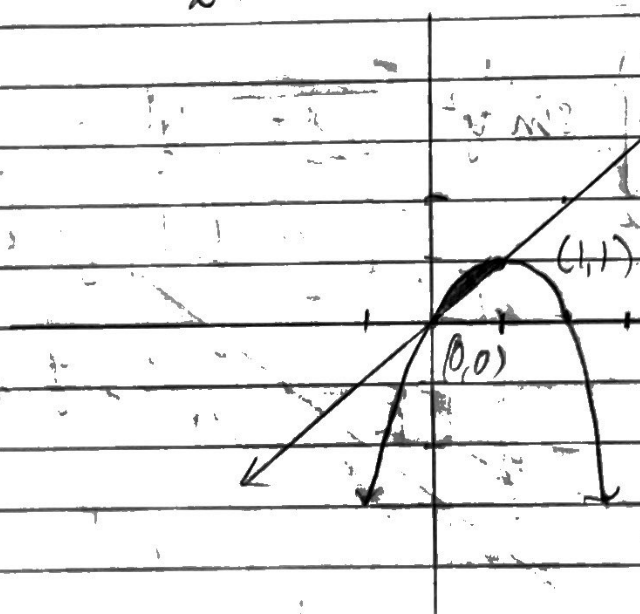
$t = y^2 \quad dt = 2y\,dy$

$$I = \frac{1}{2}\int_0^1 \sin t \, dt = \frac{1}{2}\left[-\cos t\right]_0^1 = \frac{1}{2}(1-\cos 1)$$

$$I = \frac{1-\cos 1}{2}$$

19. Evaluate $\int_0^1 \int_x^{x(2-x)} dy\,dx$ by changing order

limits



$2x - x^2$

$-(x^2 - 2x + 1) + 1 =$

$-(x-1)^2 + 1 = y$

$(x-1)^2 = -(y-1)$

$x - 1 = \pm\sqrt{-(y-1)}$

$y = 2x - x^2$ $\quad x = 1 + \sqrt{-(y-1)}$ $\times$

$x = 1 - \sqrt{1-y}$

Changing,

$$I = \int_0^1 \int_{1-\sqrt{1-y}}^{y} dx\,dy = \int_0^1 y - 1 + \sqrt{1-y} \; dy$$

$$I = \int_0^1 y\,dy - \int_0^1 dy + \int_0^1 \sqrt{1-y}\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^1 - \left[y\right]_0^1 - \left[\frac{+(1-y)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{1}{2} - 1 \mp \frac{2}{3}\left[(1-y)^{3/2}\right]_0^1$$

$$= -\frac{1}{2} + \frac{2}{3}(+1) \qquad = \cancel{-\frac{3}{2} + \frac{2}{3}} = \cancel{\frac{-9+4}{6}}$$

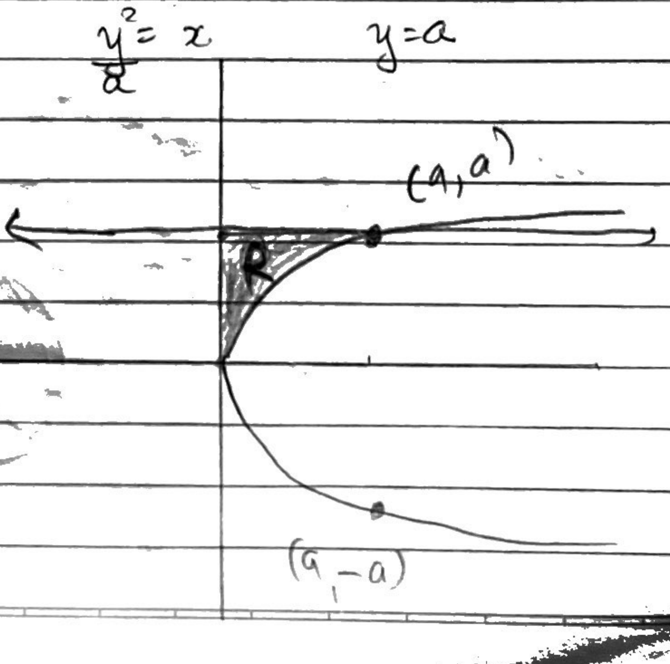
$$= \cancel{\boxed{-\frac{5}{6}}} \qquad -\frac{1}{2} + \frac{2}{3} = \frac{-3+4}{6}$$

$$= \cancel{-\frac{1}{2} - \frac{2}{3} = \frac{-3-4}{6} = \boxed{\frac{-7}{6}}}$$

$$\boxed{I = 1/6}$$

10. Change the order of integration and evaluate

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dy\,dx$$



$$I = \int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{y^4 - (ax)^2}}\,dx\,dy$$

$$I = \int_0^a \int_0^{y^2/a} \frac{y^2}{a\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\,dx\,dy$$

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$$\int_0^a \int_0^{y^2/a} \frac{y^2}{a\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\,dx\,dy$$

$$= \int_0^a \frac{y^2}{a}\left[\sin^{-1}\left(\frac{x\,a}{y^2}\right)\right]_0^{y^2/a} dy$$

$$I = \int_0^a \frac{y^2}{a}\left[\sin^{-1}\left(\frac{y^2 a}{y^2 a}\right) - \sin^{-1} 0\right] dy$$

$$\int_0^a \frac{\pi y^2}{2a} \quad \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

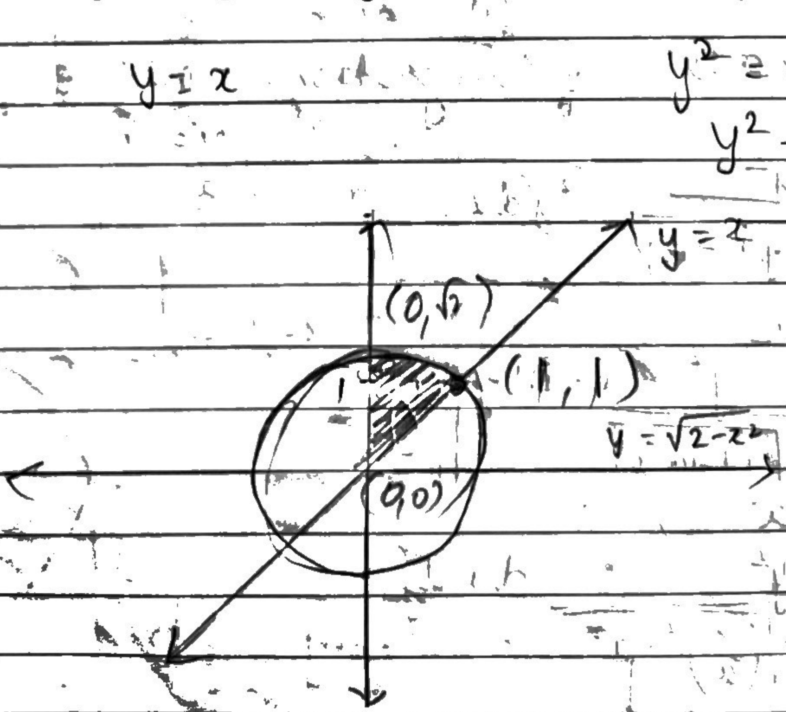
$$= \frac{\pi}{6} a^2 = \boxed{\frac{\pi a^2}{6}}$$

21. Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ by changing the order

$y = x$

$y^2 = 2 - x^2$

$y^2 + x^2 = 2$



$$\cancel{\int_0^{\sqrt{2}}\int_0^{y}} \quad I = \int_0^1\int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_0^1 \cancel{\ln\left|y+\sqrt{x^2+y^2}\right|}$$

$x^2+y^2=t$

$2x\,dx = dt$

$$I = \cancel{\int_0^1 \frac{1}{2}}$$

$2-y$

$$I = \int_0^1 \frac{1}{2}\int_{y^2}^{2y^2} \frac{dt}{\sqrt{t}}\,dy + \int_1^{\sqrt{2}}\int_{y^2}^{2} \frac{1}{2}\frac{dt}{\sqrt{t}}\,dy$$

$$= \int_0^1 \frac{1}{\cancel{2}}\left[\cancel{2}\sqrt{t}\right]_{y^2}^{2y^2} dy + \int_1^{\sqrt{2}} \frac{1}{\cancel{2}}\left[\cancel{2}\sqrt{t}\right]_{y^2}^{2} dy$$

$$= \int_0^1 (\sqrt{2}-1)y\,dy + \int_1^{\sqrt{2}} \sqrt{2} - y\,dy$$

$$= \frac{(\sqrt{2}-1)}{2} + \sqrt{2}(\sqrt{2}-1) - \left[\frac{y^2}{2}\right]_1^{\sqrt{2}}$$

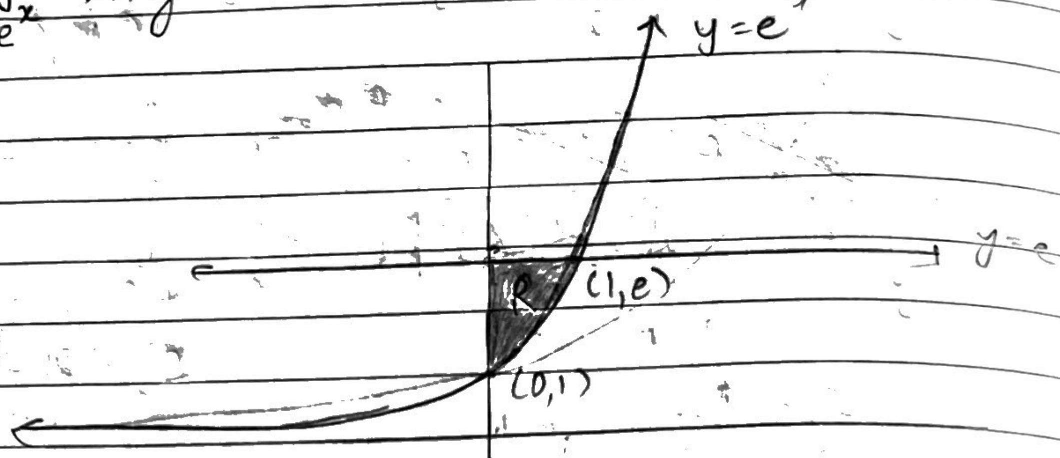
$$\frac{1}{\sqrt{2}} - \frac{1}{2} + 2 - \sqrt{2} - \left(\frac{2-1}{2}\right)$$

$$= -1 + 2 - \sqrt{2} + \frac{1}{\sqrt{2}} = 1 - \sqrt{2} + \frac{1}{\sqrt{2}}$$

$$= 1 - \frac{2+1}{\sqrt{2}} = \boxed{1 - \frac{1}{\sqrt{2}}}$$

22. Change the order of integration and evaluate

$$\int_0^1\int_{e^x}^{e} \frac{1}{\ln y}\,dy\,dx$$



$y = e^x$

$x = \ln y$

~~y = e~~

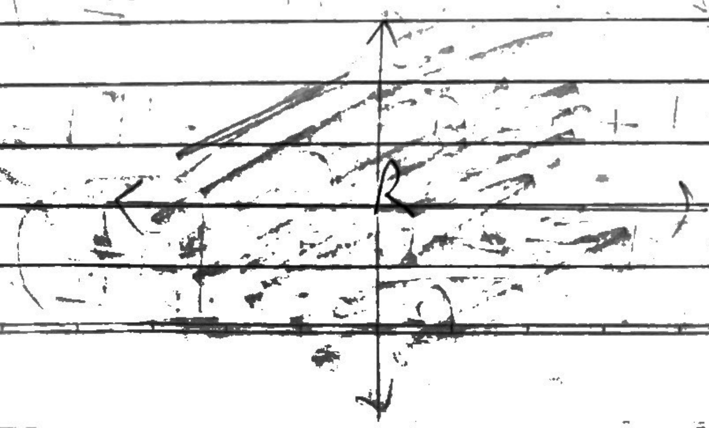
Changing order,

$$I = \int_1^e \int_0^{\ln y} \frac{1}{\ln y}\,dx\,dy$$

$$= \int_1^e \frac{1}{\ln y}\,(\ln y)\,dy = e-1 = I$$

23. Evaluate $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(1+x^2+y^2)^{3/2}}\,dx\,dy$ by changing the variables

entire x-y plane



Substituting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \frac{1}{(1+r^2)^{3/2}}\, r\,dr\,d\theta$$

$\cancel{x} \to \infty$,

$1 + r^2 = t$; $2r\,dr = dt$.

$r = 0$ $t = 1$

$$= \int_0^{2\pi} \frac{1}{2}\int_1^{\infty} \frac{dt}{t^{3/2}}\, dt\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[\frac{t^{-1/2}}{-1/2}\right]_1^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} -2\left[\frac{1}{\sqrt{t}}\right]_1^{\infty} d\theta$$

$y^2 + (x-1)^2 = 1$

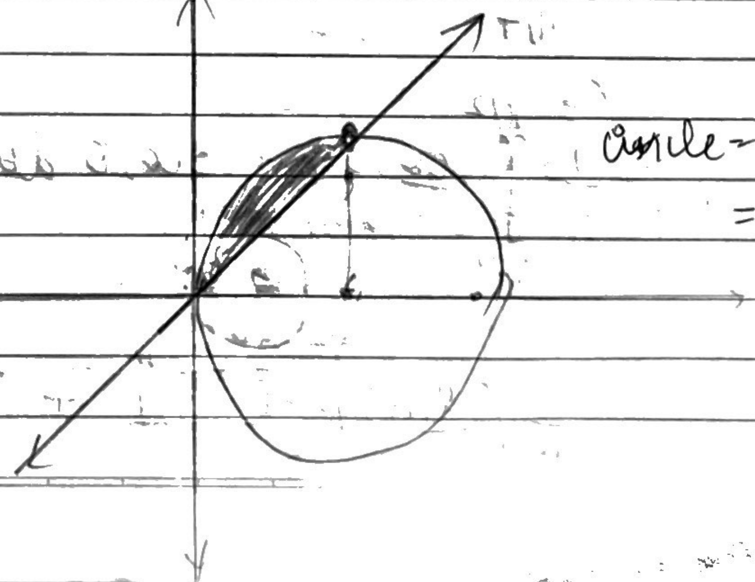
$$= -1\int_0^{2\pi} -1\, d\theta = +2\pi$$

$y^2 + x^2 - 2x = 0$

$y^2 + x^2 - 2x + 1 = 1$

24. Evaluate $\displaystyle\int_0^2\int_{x}^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$. by changing to polar coords

let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.



circle $= \cancel{2a}$

$= 2\cos\theta$

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$$I = \int_{\pi/4}^{\pi/2}\int_{0}^{2\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r}}\, r\, dr\, d\theta$$

$$I = \int_{\pi/4}^{\pi/2}\int_{0}^{2\cos\theta} r\cos\theta\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta \left(\frac{4\cos^2\theta}{2}\right) d\theta$$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$= 2\int_{\pi/4}^{\pi/2} \cos^3\theta\, d\theta$$

$\frac{\cos 3\theta + 3\cos\theta}{4}$

$$= 2\int_{\pi/4}^{\pi/2} \frac{\cos 3\theta + 3\cos\theta}{4}\, d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos 3\theta + 3\cos\theta\, d\theta = \frac{1}{2}\left(\left[\frac{\sin 3\theta}{3}\right]_{\pi/4}^{\pi/2} + \left[3\sin\theta\right]_{\pi/4}^{\pi/2}\right)$$

$$= \frac{1}{2}\left(\frac{\sin 3\pi/2 - \sin 3\pi/4}{3} + 3(\sin \pi/2 - \sin \pi/4)\right)$$

$$= \frac{1}{2}\left(\frac{-1+1/\sqrt{2}}{3} + 3\left(1-\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{+1}{6}\left(-1+\frac{1}{\sqrt{2}}\right) + \frac{3}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

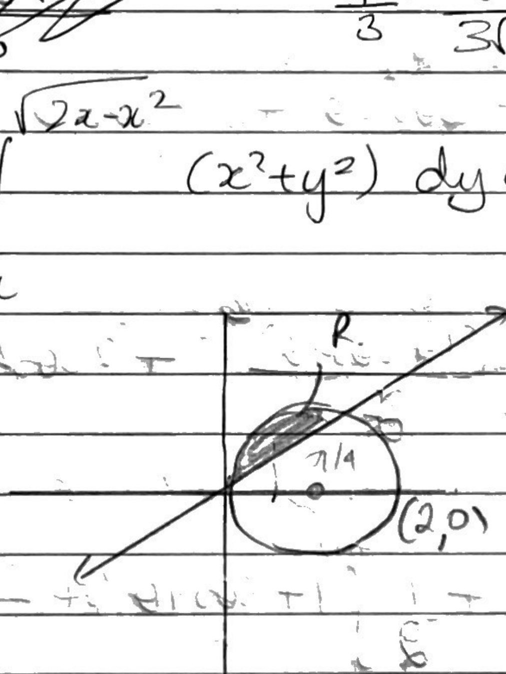
$$= \frac{-1}{6} - \frac{1}{6\sqrt{2}} + \frac{3}{2} - \frac{3}{2\sqrt{2}}$$

$$= \frac{-1}{6} + \frac{9}{6} - \frac{1}{6\sqrt{2}} - \frac{9}{6\sqrt{2}}$$

$$= \frac{4}{3} - \frac{10}{6\sqrt{2}} = \cancel{\frac{5}{3}} - \cancel{\frac{10\sqrt{2}}{3\times 2}} = \cancel{5-5}$$

$$= \cancel{\frac{10-5\sqrt{2}}{6}} = \frac{4}{3} - \frac{5}{3\sqrt{2}} = \frac{4}{3} - \frac{5\sqrt{2}}{3}$$

25 Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ by changing to polar.



Let
$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$.

$r = 2\cos\theta$.

$$= \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{4}\times 16\cos^4\theta\,d\theta$$

$$\cancel{= 4\int_{\pi/4}^{\pi/2} \cos^3\theta(1-\sin^2\theta)^{3/2}\,d\theta}$$

$\cancel{\sin\theta = t}$
$\cancel{\cos\theta\,d\theta = dt}$

$$\cancel{= 4\int_{1/\sqrt{2}}^{1} (1-t^2)^{3/2}\,dt}$$

2/6

$$= \cancel{4\int_{\pi/4}^{\pi/2} \cos^2\theta\,(1-\sin^2\theta)\,d\theta}$$

$$= \cancel{4\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta} - 4\int_{\pi/4}^{\pi/2} \cancel{\cos^2\theta\,\sin^2\theta}\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2} \cos^4\theta\,d\theta = \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)^2\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} 1 + \cos^2 2\theta + 2\cos 2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} 1 + \frac{(1+\cos 4\theta)}{2} + 2\cos 2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} 1+\cos 4\theta\,d\theta + 2\int_{\pi/4}^{\pi/2} \cos 2\theta\,d\theta$$

$$= \frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos 4\theta\,d\theta + 2\int_{\pi/4}^{\pi/2} \cos 2\theta\,d\theta$$

$$= \frac{\pi}{4} + \frac{\pi}{8} + \frac{1}{2}\left[\frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2} + \frac{2}{2}\left[\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

store 67 /219

$= \frac{3\pi}{8} + \frac{1}{8}(0) + \sin\pi - \sin\pi/2$

$= -\cancel{\frac{4\pi}{8}} \quad \boxed{\frac{3\pi}{8} - 1}$

Q6. $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$

$x = r\cos\theta$

$y = r\sin\theta$

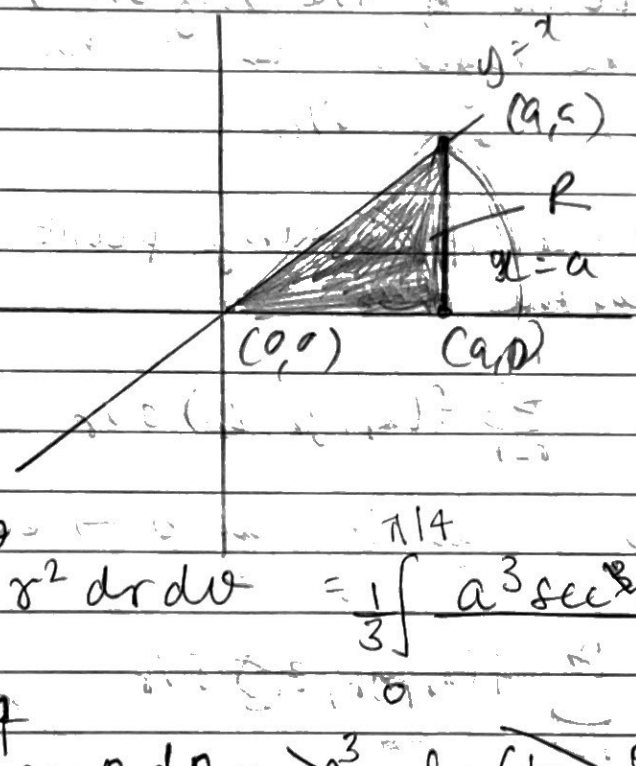
$dx\,dy = r\,dr\,d\theta$

$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$

$x = a$

$r\cos\theta = a$

$r = a\sec\theta.$



$I = \int_0^{\pi/4} \cos^2\theta \int_0^{a\sec\theta} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/4} a^3\sec^3\theta\,.\cancel{\cos^2\theta}\,d\theta$

$= \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\,d\theta = \cancel{\frac{a^3}{3}\ln\left(\tan\frac{\theta}{2}+\frac{\pi}{4}\right)\Big|_0^{\pi/4}}$

$= \frac{a^3}{3}\cancel{\ln\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\right) - \ln(\tan\pi/4)}$

$\cancel{\frac{a^3}{3}\ln(}$

$$= \frac{a^3}{3} \int_0^{\pi/4} \sec\theta \, d\theta = \frac{a^3}{3} \left[\ln(\sec\theta + \tan\theta) \right]_0^{\pi/4}$$

$$= \frac{a^3}{3} \left[\ln\left(\sqrt{2}+1\right) - \ln(1) \right]$$

$$= \frac{a^3}{3} \ln(\sqrt{2}+1)$$

09.10.19

TRIPLE INTEGRALS

let $f(x,y,z)$ be a continuous function defined at every point of the region V in 3D space. Divide the region into n elementary volumes $\delta V_1, \delta V_2, \delta V_3 \dots \delta V_n$.

let (x_r, y_r, z_r) be any point in δV_r. Consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r$$

Taking the limit as $n \to \infty$. If

$$\lim_{\substack{n \to \infty \\ (\delta V_r \to 0)}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r$$

exists uniquely and finitely, it is called the triple integral of $f(x,y,z)$ over the region V, written as

$$\iiint_V f(x,y,z)\, dV$$

Convention & Language

$$\int_{x=a}^{b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz$$

outermost

middle/second

innermost integral

Shifting coordinates.

1. Cylindrical coordinate system
2. Spherical coordinate system.

27. Evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} x+y+z \; dy\,dx\,dz$

$$I = \int_{-1}^{1}\int_{0}^{z}\left[xy+zy+\frac{y^2}{2}\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} x(x+z)+z(x+z)+\frac{(x+z)^2}{2}-x(x-z)-z(x-z)-\frac{(x-z)^2}{2}\,dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \cancel{x^2}+xz+xz+z^2+\frac{(x+z)^2-(x-z)^2}{2}-\cancel{x^2}+\cancel{zx}-\cancel{xz}+z^2$$

$$= \int_{-1}^{1}\int_{0}^{z} 4xz+2z^2\,dx\,dz = \int_{-1}^{1}\left[\frac{4zx^2}{2}+2z^2x\right]_{0}^{z} dz$$

$$= \int_{-1}^{1} \frac{4z^3}{2} + 2z^3 \, dz = \int_{-1}^{1} 4z^3 \, dz$$

$\because$ it is an odd function from $-a$ to a

$\therefore I = 0$.

28. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dz \, dy \, dx$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}} \right) \right]_0^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} - 0 \, dy \, dx$$

$$= \int_0^1 \frac{\pi}{2} \sqrt{1-x^2} \, dx = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} \, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1^2}{2} \sin^{-1}(x) \right]_0^1$$

$$= \frac{\pi}{2} \left(0 + \frac{1}{2} \sin^{-1}(1) - 0 \right)$$

$$= \frac{\pi}{2} \times \frac{1}{2} \times \frac{\pi}{2} = \boxed{\frac{\pi^2}{8}}$$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3$. store 67 sub-int - 223 1 - diff

29. $$\int_0^{\ln 2}\int_0^{x}\int_0^{x+\ln y} e^{x+z+y}\,dz\,dy\,dx.$$

$$I = \int_0^{\ln 2}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+\ln y} dy\,dx$$

$$= \int_0^{\ln 2}\int_0^{x} e^{x+y+x+\ln y} - e^{x+y}\,dy\,dx$$

$$= \int_0^{\ln 2}\int_0^{x} e^{2x+y+\ln y} - e^{x+y}\,dy\,dx$$

$$= \int_0^{\ln 2}\int_0^{x} e^{2x+y+\ln y}\,dy - \int_0^{x} e^{x+y}\,dy\,dx$$

$$= \int_0^{\ln 2}\int_0^{x} e^{2x}ye^{y}\,dy - \int_0^{x} e^{x}\cdot e^{y}\,dy\quad dx.$$

Let $I_1 = \int_0 \overset{u}{y}\,\overset{v}{e^{y}}\,dy = ye^{y} - e^{y}$

$$I = \int_0^{\ln 2} e^{2x}\left[ye^{y}-e^{y}\right]_0^{x} - e^{x}\left[e^{y}\right]_0^{x}\,dx$$

$$= \int_0^{\ln 2} e^{2x}\left(2xe^{\cancel{2x}} - \cancel{e^{2x}+1}\right) - \cancel{e^{2x}} + e^{x}\,dx$$

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$$\int_0^{\ln 2} e^{2x}\left[ye^{y}-e^{y}\right]_0^{x} - e^{2x} + e^{x}\,dx$$

$$I = \int_0^{\ln 2} e^{2x}\left(xe^{x}-e^{x}+1\right) - e^{2x} + e^{x}\,dx$$

$$= \int_0^{\ln 2} xe^{3x} - e^{3x} + \cancel{e^{2x}} - \cancel{e^{2x}} + e^{x}\,dx$$

$$I_2 = \int x \cdot e^{3x} = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

$$= \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right]_0^{\ln 2} - \left[\frac{e^{3x}}{3}\right]_0^{\ln 2}$$

$$\cancel{+\left[\frac{e^{2x}}{2}\right]_0^{\ln 2}} + \left[e^{x}\right]_0^{\ln 2} + \cancel{[x]_0^{\ln 2}}$$

$$I = \left(\frac{\ln 2\, e^{3\ln 2}}{3} - \frac{e^{3\ln 2}}{9} + \frac{1}{9}\right) - \frac{e^{3\ln 2}}{3} + \frac{1}{3}$$

$$\cancel{+\frac{e^{2\ln 2}}{2}} - \cancel{\frac{1}{2}} - \cancel{\ln 2} + 1 + \cancel{\ln 2}$$

$$= \frac{\ln 2 \times 8}{3} - \frac{8}{9} + \frac{1}{9} - \frac{8}{3} + \frac{1}{3} + 2 - \frac{1}{2} + 1$$

$$= \frac{8}{3}\ln 2 - \frac{7}{9} - \frac{7}{3} + 3 + \frac{1}{2}$$

$= \frac{\ln 2 \times 8}{3} - \frac{8}{9} + \frac{1}{9} - \frac{8}{3} + \frac{1}{3} + 2 - 1$

$= \frac{8}{3}\ln 2 - \frac{7}{9} - \frac{7}{3} + \cancel{\frac{2}{3}} = 1$

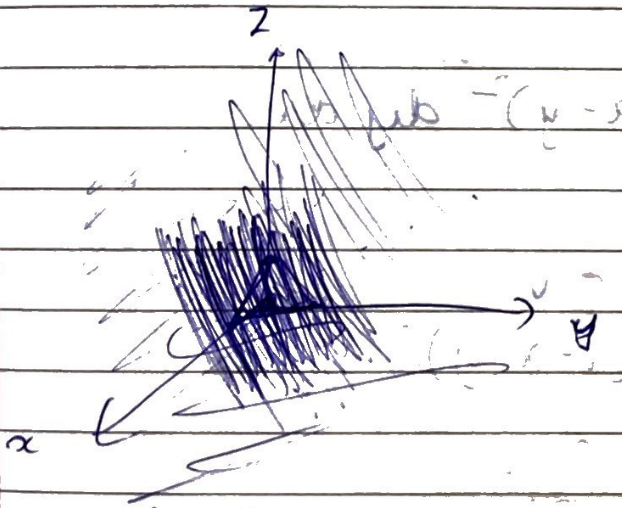
$= \frac{8}{3}\ln 2 - \frac{7}{9} - \frac{7}{3} \cancel{- \frac{8\ln 2}{3}} + 1$

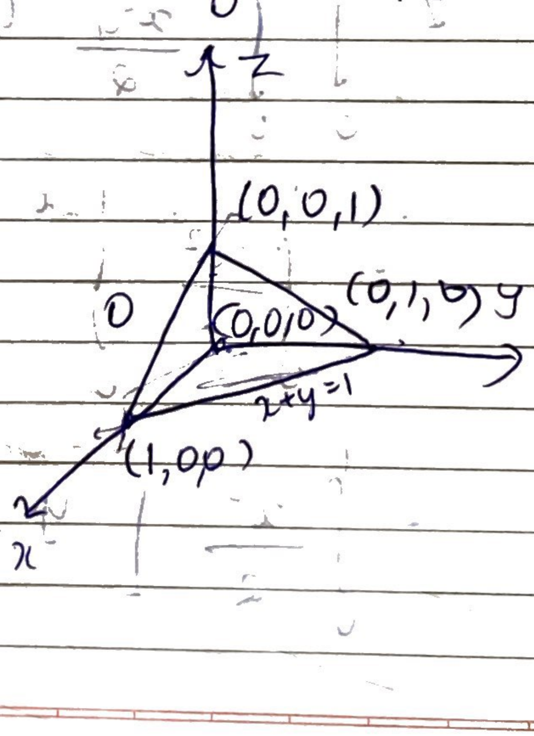
$= \frac{8}{3}\ln 2 - \frac{7}{9} - \frac{21}{9} + \frac{9}{9} = \frac{8}{3}\ln 2 - \frac{19}{9}$

* watch spherical, polar coordinates

30. $\iiint x^2yz \, dx\, dy\, dz$ over the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

triangular prism





z-direction
$z=0$ to $x+y+z=1$

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$$I = \iiint_R x^2 y z \, dz\, dy\, dx.$$

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} x^2 y z \, dz\, dy\, dx$$

$$I = \int_0^1 \int_0^{1-x} \left[\frac{x^2 y z^2}{2}\right]_0^{1-x-y} dy\, dx$$

$$I = \int_0^1 \int_0^{1-x} \frac{x^2 y}{2} (1-x-y)^2 \, dy\, dx$$

$$I = \int_0^1 \int_0^{1-x} \frac{x^2 y}{2} (1-x-y)^2 \, dy\, dx$$

$$= \int_0^1 \frac{x^2}{2} \int_0^{1-x} \overset{u}{y} \, \overset{v}{(1-x-y)^2} \, dy\, dx$$

$$= \int_0^1 \frac{x^2}{2} \left[y \frac{(1-x-y)^3}{-3} - \frac{(1-x-y)^4}{12} \right]_0^{1-x} dx$$

$$I = \int_0^1 \frac{x^2}{2}\left((1-x)(0) - (0) - 0 + \frac{(1-x)^4}{12}\right) dx$$

$$I = \int_0^1 \frac{x^2}{2}\,\frac{(1-x)^4}{12}\, dx = \frac{1}{24}\int_0^1 x^2 (1-x)^4\, dx$$

$$u = x^2 \qquad v = (1-x)^4$$

$$I = \frac{1}{24}\left[-2x\,\frac{(1-x)^5}{-5} + \frac{2\,(1-x)^6}{30}\right]_0^1 + x^2$$

$$= \frac{1}{24}\left(\frac{2\,(1-0)^6}{30}\right) = \frac{1}{15} \times \frac{1}{24} = \frac{1}{30} \times \frac{1}{12}$$

$$I = \frac{1}{360}.$$

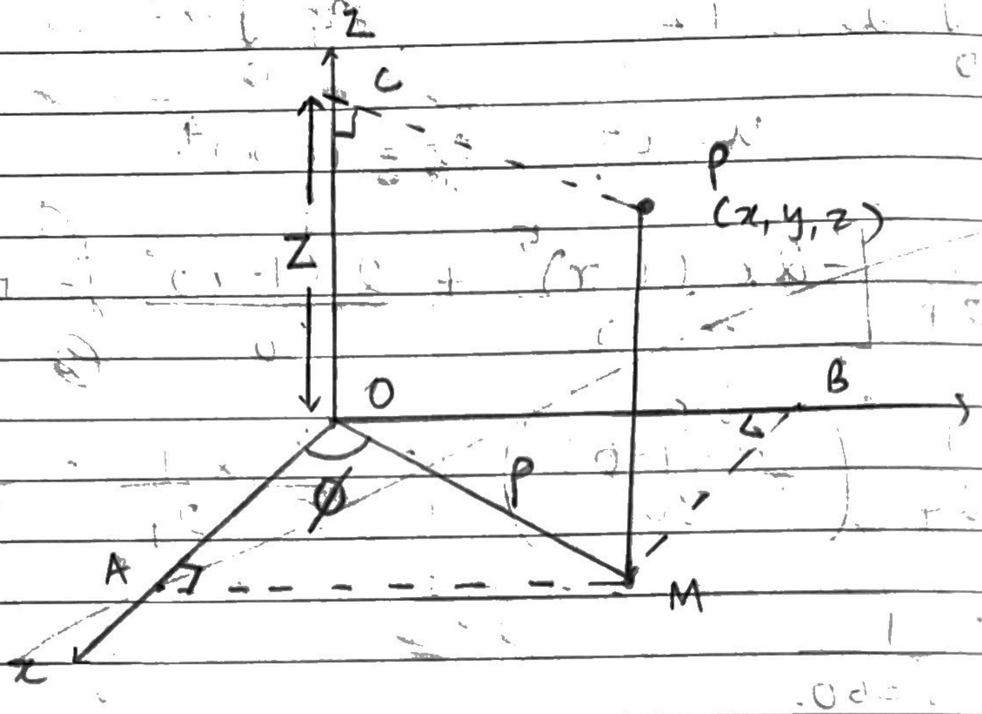
$$I = \frac{1}{24}\left[x^2\,\frac{(1-x)^5}{-5} - \frac{2x\,(1-x)^6}{30} + \frac{2\,(1-x)^7}{-7\times 30}\right]_0^1$$

$$= \frac{1}{\cancel{24}_{12}}\left(\frac{+\cancel{2}}{+7\times 30}\right) = \frac{1}{12\times 7\times 30} = \frac{1}{2520}$$

$$\boxed{I = \frac{1}{2520}}$$

Changing of Variables – Evaluation of Triple Int

CYLINDRICAL COORDINATES (POLAR)



- Let $P(x, y, z)$ be any point in 3D space.
- Draw $PM \perp XOY$ plane. Join OM.
- Let $OM = \rho$

  $\angle XOM = \phi$

  $PM = Z$

From the figure,

$$\boxed{\begin{aligned} x &= OA = \rho\cos\phi \\ y &= OB = \rho\sin\phi \\ z &= OC = Z. \end{aligned}}$$

the numbers (ρ, ϕ, z) are called cylindrical polar coordinates of P, where $\rho \geq 0$, $0 \leq \phi \leq 2\pi$ and $-\infty < z < \infty$

- For points on z-axis, $\rho = 0$

- $\phi = 0$ for all points on x-z plane.
For points on x-axis, $z = 0$ and $\phi = 0$.

- $z = 0$ for all points on x-y plane.
For points on y-axis, $z = 0$ and $\phi = \pm\pi/2$

If the distance ρ is kept constant, then the locus of P is a cylinder.

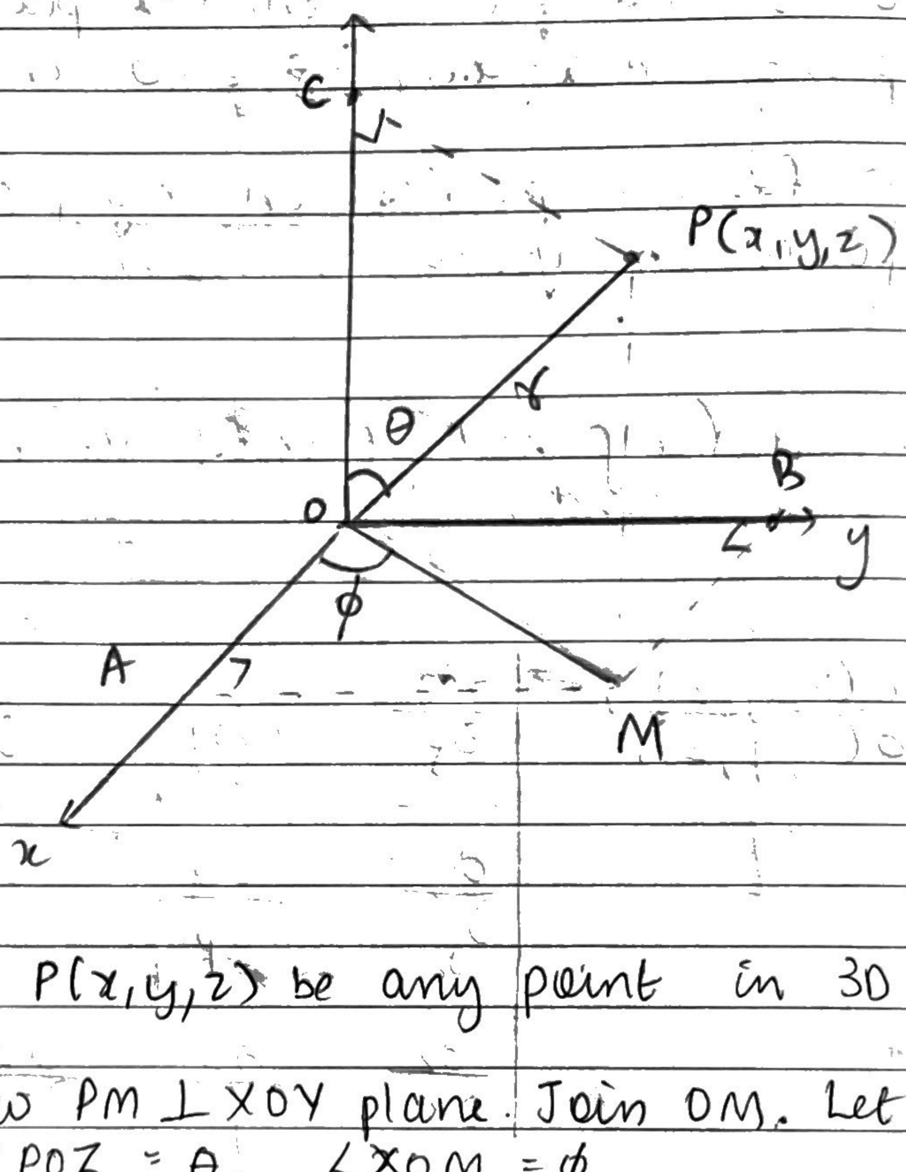
$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho(\cos^2\phi + \sin^2\phi)$$

$$\boxed{J = \rho}$$

$\therefore$ $dV = dx\,dy\,dz$ has to be replaced by $\rho\, d\rho\, d\phi\, dz$.

SPHERICAL POLAR COORDINATES



Let $P(x, y, z)$ be any point in 3D space.

Draw $PM \perp XOY$ plane. Join OM. Let $OP = r$, $\angle POZ = \theta$, $\angle XOM = \phi$.

From the figure,

$PC = r\sin\theta = OM$

$OC = r\cos\theta$

$$x = OM\cos\phi = r\sin\theta\cos\phi$$

$$y = OM\sin\phi = r\sin\theta\sin\phi$$

$$z = OC = r\cos\theta$$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

$$x^2+y^2+z^2 = r^2$$

The numbers (r, θ, ϕ) are called spherical polar coordinates of P, where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$

If the distance r is kept constant, then the locus of P is a sphere whose equation is $x^2+y^2+z^2 = r^2$.

If the angle θ is kept constant, then the locus of P is a cone.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left(r^2\cos\theta\cos^2\phi\sin\theta + r^2\sin\theta\cos\theta\sin^2\phi\right)$$
$$+ r\sin\theta\left(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right)$$

$$= \cos\theta\left(r^2\cos\theta\sin\theta\right) + r\sin\theta\left(r\sin^2\theta\right)$$

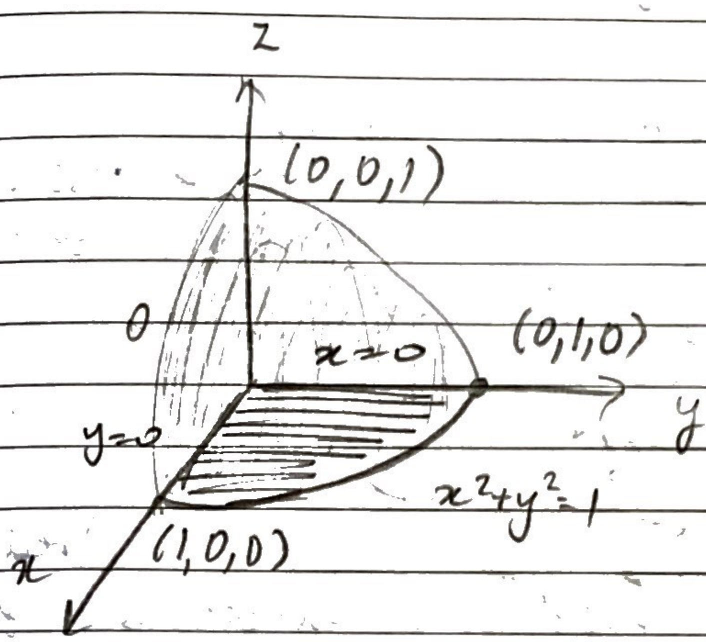
$$= r^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$\boxed{J = r^2\sin\theta}$$

$\therefore$ $dV = dx\,dy\,dz$ has to be changed to $r^2\sin\theta\, dr\, d\theta\, d\phi$.

31. Evaluate $\iiint \frac{dV}{\sqrt{1-x^2-y^2-z^2}}$ over the sphere $x^2+y^2+z^2=1$ in the positive octant.

(refer pg 222, problem 28)



Shifting to polar coordinates.

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{\sqrt{1-r^2}}$$

$$= \int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=\pi/2} + \sin\theta \int_{r=0}^{r=1} \frac{+r^2 \, dr}{\sqrt{1-r^2}} \, d\theta \, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \cancel{-\sin\theta \int \frac{1-r^2 \, dr}{\sqrt{1-r^2}}} \quad \cancel{+\sin\theta} \int \frac{dr}{\sqrt{1-r^2}} \quad d\theta \, d\phi$$

Let $r = \sin t$ $\quad dr = \cos t \, dt$

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \int_0^{\pi/2} \frac{t\sin^2 t\,\cancel{\cos t}\,dt}{\cancel{\cos t}}\, d\theta\, d\phi$$

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\sin\theta\int_0^{\pi/2} t\sin^2 t\, dt\quad d\theta\, d\phi$$

$\cos 2x = 1 - 2\sin^2 t$

$\frac{\cos 2x - 1}{2}$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\sin\theta\int_0^{\pi/2}$$

Break into product of integrals | red. formula page 20?

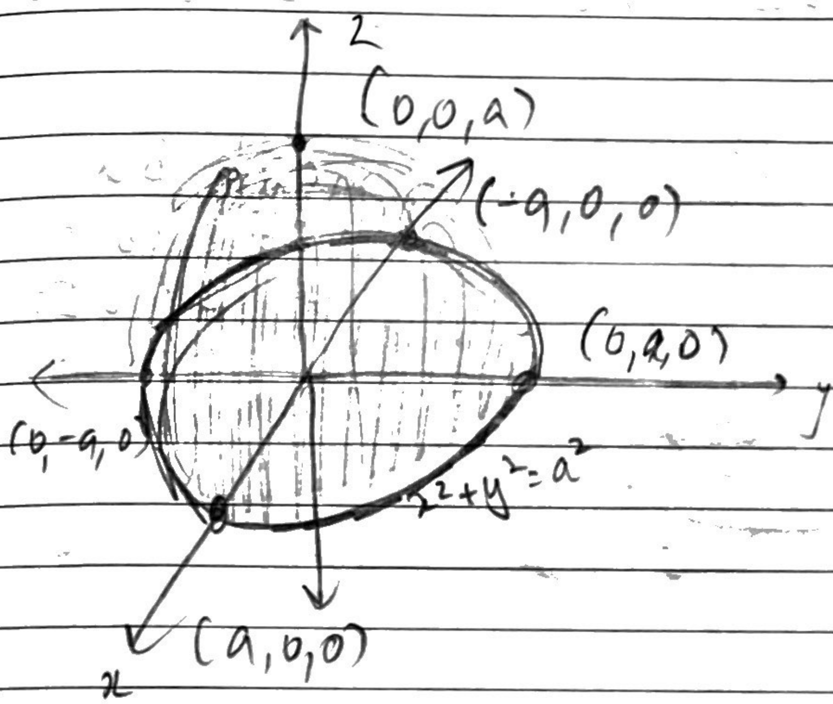
$$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} t\sin^2 t\, dt$$

$$= \left(\frac{\pi}{2}\right)\Big[-\cos\theta\Big]_0^{\pi/2} + \left(\frac{2-1}{2}\right)\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{\pi}{2}\right)(1)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8}$$

$$\boxed{I = \frac{\pi^2}{8}}$$

32. Find the value of $\iiint z\, dV$ over the hemisphere $x^2+y^2+z^2 \le a^2$ and $z \ge 0$



r : 0 to a
θ = 0 to $\pi/2$
ϕ : 0 to 2π

$z = r\cos\theta$
$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

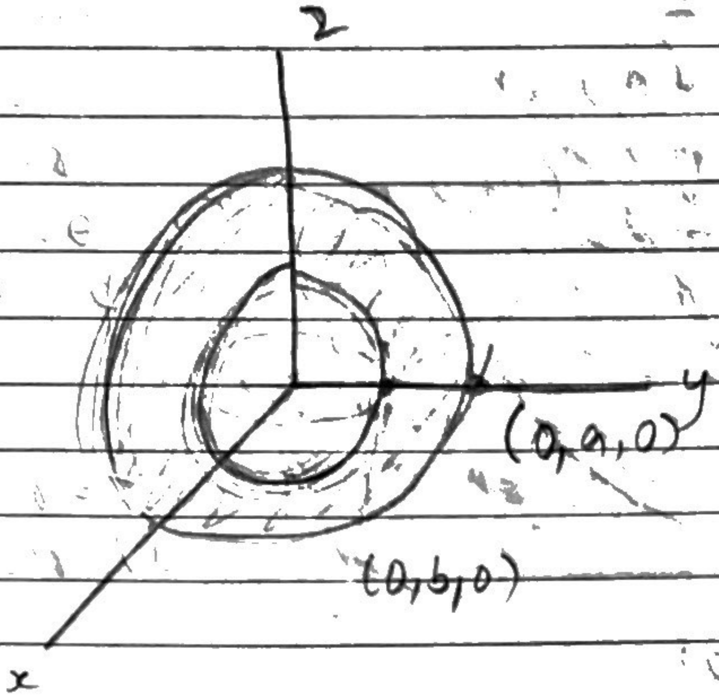
$$I = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=a} r\cos\theta\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/2} \underline{\sin\theta\cos\theta} \int_{r=0}^{r=a} r^3\, dr\, d\theta\, d\phi$$

$$= (2\pi) \int_0^1 t\, dt \int_0^a r^3\, dr = (2\pi)\left(\frac{1}{2}\right)\left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{2\pi a^4}{2\times 4} = \frac{\pi a^4}{2\times 2} = \boxed{\frac{\pi a^4}{4}}$$

33. Evaluate $\iiint \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$ over the region bounded by the sphere $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ where $a>b>0$



r : b to a

$\theta = 0$ to π

$\phi = 0$ to 2π

$$I = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=b}^{r=a} \frac{r^2 \sin\theta}{r}\, dr\, d\theta\, d\phi$$

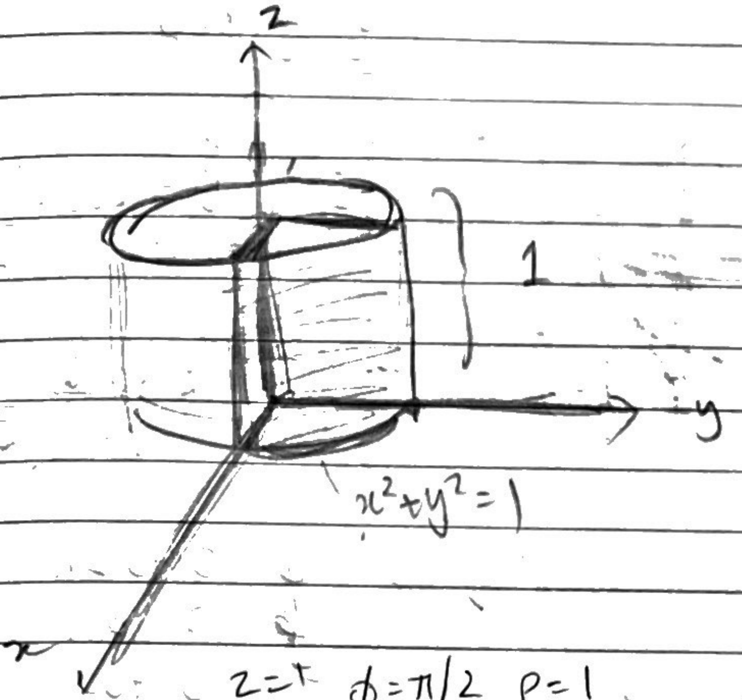
$$= (2\pi)\int_0^{\pi} \sin\theta\, d\theta \int_b^a r\, dr$$

$$= (2\pi)(2)\left[\frac{r^2}{2}\right]_b^a = \frac{4\pi}{2}(a^2-b^2)$$

$$\boxed{I = 2\pi(a^2-b^2)}$$

34. If R is the region bounded by the planes $x=0$, $y=0$, $z=0$, $z=1$ and $x^2+y^2=1$ ← cylinder in 3D

Evaluate $\iiint_R xyz \, dx\,dy\,dz$ by changing to cylindrical coordinates



$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$

$\rho = 0$ to 1
$\phi = 0$ to $\pi/2$
$z = 0$ to 1

$J = \rho$.

$$I = \int_{z=0}^{z=1}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=1} \rho^2\cos\phi\sin\phi \, z \, \rho \, d\rho \, d\phi \, dz$$

$$= \int_0^1 z\,dz \int_0^{\pi/2} \cos\phi\sin\phi \, d\phi \int_0^1 \rho^3 \, d\rho$$

$$= \left(\frac{1}{2}\right)\left(\int_0^1 t\,dt\right)\left[\frac{\rho^4}{4}\right]_0^1$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \boxed{\frac{1}{16} = I}$$

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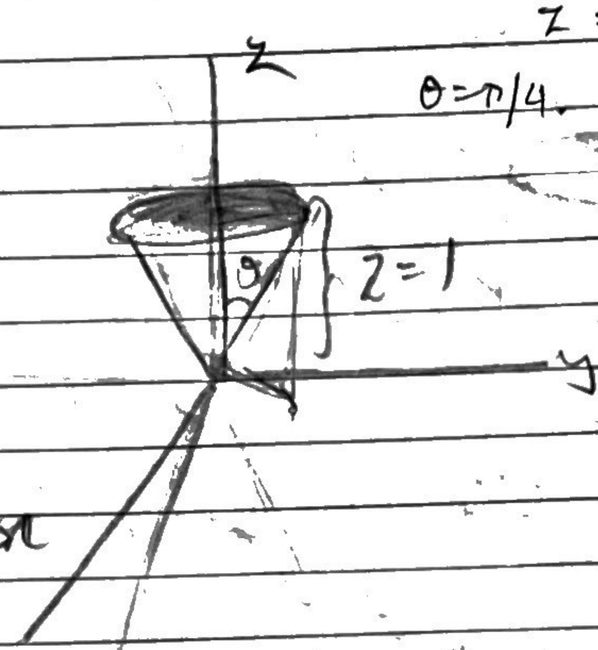
35. Evaluate $\iiint_R \sqrt{x^2+y^2}\, dx\,dy\,dz$ where the region bounded by $z=0,\ z=1,\ x^2+y^2=z^2$ — cone

Using cylindrical polar coordinates

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$



$\theta = \pi/4$. $\theta = \tan^{-1}\frac{r}{z} = \frac{\pi}{4}$

~~$\tan\theta = \frac{r}{z}$ $r = z$~~

$\phi : 0$ to 2π

$\rho : 0$ to 1

$z = 0$ to 1

$$\cancel{I = \int_{z=0}^{z=1}\int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=1} \rho^2\, d\rho\, d\phi\, dz}$$

$$\cancel{= \int_0^1 dz \int_0^{2\pi} d\phi \int_0^1 \rho^2 d\rho = (1)(}$$

$x^2+y^2 = z^2 = \rho^2$

$$= \int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=\rho}^{z=1} \rho^2\, d\rho\, d\phi\, dz$$

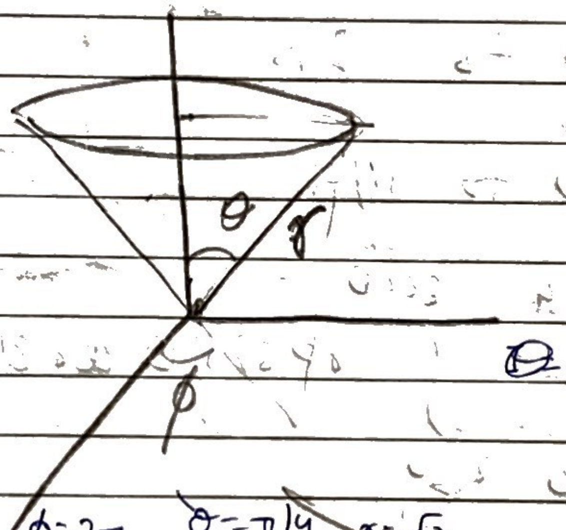
$$I = \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=1} \rho^2 (1-\rho)\, d\rho\, dz$$

$$= \int_0^{2\pi} \int_0^1 \rho^2 - \rho^3\, d\rho\, dz = (2\pi) \int_0^1 \rho^2 - \rho^3\, d\rho$$

$$= 2\pi \left[\frac{\rho^3}{3} - \frac{\rho^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{1}{12}\right) = \boxed{\frac{\pi}{6}}$$

<u>using Spherical coordinates</u>



$r = 0$ to $\sqrt{2}$

$\phi = 0$ to 2π

$\theta = 0$ to $\pi/4$

$J = r^2 \sin\theta$

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$$I = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sqrt{2}} r^2 \sin\theta\; r\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/4} \sin^2\theta\, d\theta \int_0^{\sqrt{2}} r^3\, dr$$

$$(2\pi) \int_0^{\pi/4} \frac{\cos 2\theta - 1}{2} \, d\theta \int_0^{\Omega} r^3 \, dr$$

$$I = (2\pi)\left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \int_0^{\pi/4} \frac{\cos 2\theta}{2} - \frac{1}{2} \, d\theta$$

$$= (2\pi)\left(\frac{4}{4}\right)\left(\frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} - \frac{1}{2}\left(\frac{\pi}{4}\right)\right)$$

$$= (\pi)\left(\frac{1}{2} - \frac{\pi}{4}\right)$$

r: 0 to $1/\cos\theta$

ϕ : 0 to 2π

θ : 0 to $\pi/4$

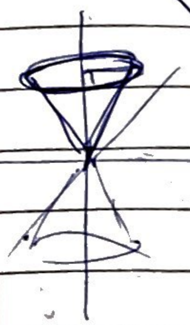
$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{r=0}^{\sec\theta} r^3 \sin^2\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2} (2\pi) \int_0^{\pi/4} \sin^2\theta \left[\frac{r^4}{4}\right]_0^{\sec\theta} d\theta$$

$$= \frac{(2\pi)}{4} \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta} \sec^2\theta \, d\theta = \frac{\pi}{2} \int_0^{\pi/4} \tan^2\theta \sec^2\theta \, d\theta$$

$$I = \left[\frac{\tan^3\theta}{3}\right]_0^{\pi/4} \left(\frac{\pi}{2}\right) = \left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

36. Evaluate $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^1 \sqrt{x^2+y^2}\, dV$ using cylindrical coordinates



~~$x = r\cos\theta$~~

$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$
$dx\,dy\,dz \to \rho\, d\rho\, d\phi\, dz$

$I = \int_0^1 \int_0^{\sin^{-1}(1/\rho)}$

$z : \sqrt{x^2+y^2}$ to 1
$z = \rho$ to 1

$y : 0$ to 1

$\rho :$

$$\int_0^{2\pi}\int_0^1\int_\rho^1 \rho^2\, dz\, d\rho\, d\phi$$

$$(2\pi)\int_0^1\int_\rho^1 \rho^2\, dz\, d\rho = 2\pi\int_0^1 \rho^2(1-\rho)\, d\rho$$

$$= 2\pi\int_0^1 \rho^2-\rho^3\, d\rho = 2\pi\left[\frac{1}{3}-\frac{1}{4}\right] = 2\pi\left(\frac{1}{12}\right) = \frac{\pi}{6}$$

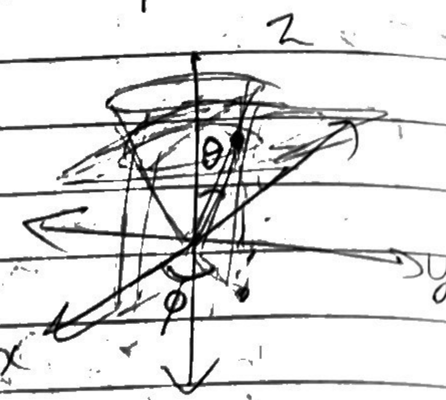
36 Use spherical polar coordinates to evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$

$z^2 = x^2 + y^2$

$\cancel{x = r\cos\ r\sin\theta\cos\phi}$

$\cancel{y =}$

$\cancel{z = r\cos\theta}$



The eq. $z = \sqrt{x^2+y^2}$

$r\cos\theta = r\sin\theta$

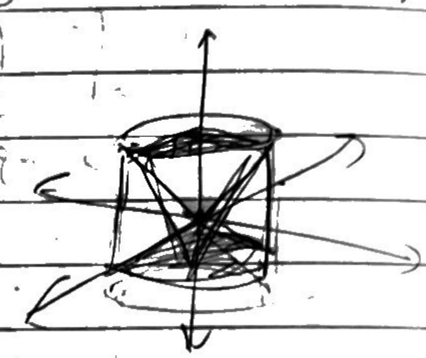
$\tan\theta = 1$

$\theta = \pi/4$ → cone with ~~vertex~~ at origin ~~with semi~~ vertical angle $\pi/4$

$z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$



eq $z = 1$

$r\cos\theta = 1$

$r = \sec\theta$

the eq. $y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$ is a cylinder with centre $(0,0)$, radius 1

ϕ varies from $0, \cancel{\pi/2}$ to $\pi/2$

$J = r^2 \sin\theta$

ϕ varies from 0 to $\pi/4$.

store 67

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/4}\int_{r=0}^{\sec\theta} \frac{r^2\sin\theta}{r}\, dr\, d\theta\, d\phi$$

$$I = \left(\frac{\pi}{2}\right)\int_{0}^{\pi/4}\int_{r=0}^{\sec\theta} r\sin\theta \, dr\, d\theta$$

$$= \frac{\pi}{2}\int_{0}^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_0^{\sec\theta} d\theta$$

$$= \frac{-\pi}{2}\int_{0}^{\pi/2} \frac{-\sin\theta}{2\cos^2\theta}\, d\theta$$

$t = \cos\theta$

$dt = -\sin\theta \, d\theta$

$$= \frac{-\pi}{4}\int_{1}^{0} \frac{dt}{t^2}\, dt$$

$\theta = 0 \to t = 1$

$\theta = \pi/2 \to t = 0$

$$= \left(-\frac{\pi}{4}\right)\Big($$

$$= \frac{\pi}{4}\int_{0}^{\pi/4} \tan\theta\sec\theta \, d\theta = \left(\sec\frac{\pi}{4} - \sec 0\right)\frac{\pi}{4}$$

$$\boxed{I = \left(\frac{\pi}{4}\right)\left(\sqrt{2}-1\right)}$$

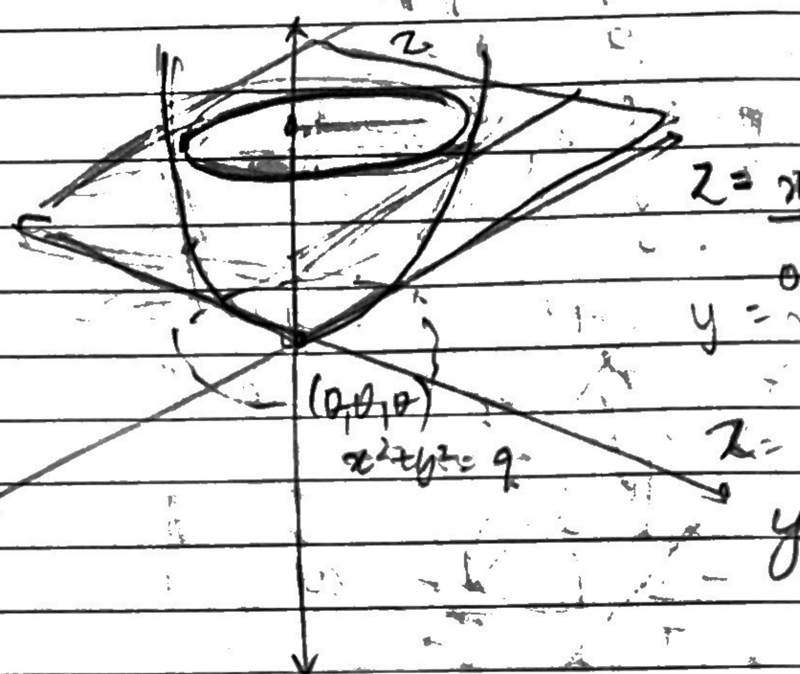
37. Evaluate $\iiint_R (x^2+y^2)\, dx\, dy\, dz$

over the region bounded by the paraboloid $x^2+y^2=3z$ and the plane $z=3$

$x^2+y^2=az$

↓

paraboloid.



$z = \frac{x^2+y^2}{3}$ to 0

0 to 3

$y = \sqrt{9-x^2}$

$z = 0$ to 3

Using cylindrical coordinates:

$$\rho^2\cos^2\phi + \rho^2\sin^2\phi = \cancel{3z}$$

$$\boxed{\frac{\rho^2}{\cancel{3}}}$$

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\int_{z=\frac{\rho^2}{3}}^{3} \frac{\rho^2}{\cancel{3}}\,\rho\,\cancel{\ }\; dz\, d\rho\, d\phi$$

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{3} \frac{\cancel{\ }}{3}\left[\frac{\rho^4}{4}\right]_{\frac{\rho^2}{3}}^{3} d\rho\, d\phi$$

store 67 — 12.4.5

$$\frac{2\pi}{4\times 3}\int_{\rho=0}^{3}\left[\rho^4\right]_{\rho^2/3}^{3} d\rho$$

$$= \left(\frac{\pi}{6}\right)\int_{0}^{3} 3^4 - \frac{\rho^8}{3^4}\, d\rho$$

$$\left(\frac{\pi}{6}\right)\left[3^5 - \frac{3^9}{9\times 3^4}\right]$$

$$\left(\frac{\pi}{6}\right)\left(3^5 - \frac{3^5}{9}\right) = \frac{3^5\pi}{\cancel{6}_{3}}\left(\frac{\cancel{8}^{4}}{9}\right) = 3\pi\times 4$$

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\int_{z=\rho^2/3}^{3} \rho^3\, dz\, d\rho\, d\phi$$

$$\int_{0}^{2\pi}\int_{0}^{3}\left(3-\frac{\rho^2}{3}\right)\rho^3\, d\rho\, d\phi$$

$$(2\pi)\int_{0}^{3} 3\rho^3 - \frac{\rho^5}{3}\, d\rho = 2\pi\left[\frac{3\rho^4}{4} - \frac{\rho^6}{6\times 3}\right]_{0}^{3}$$

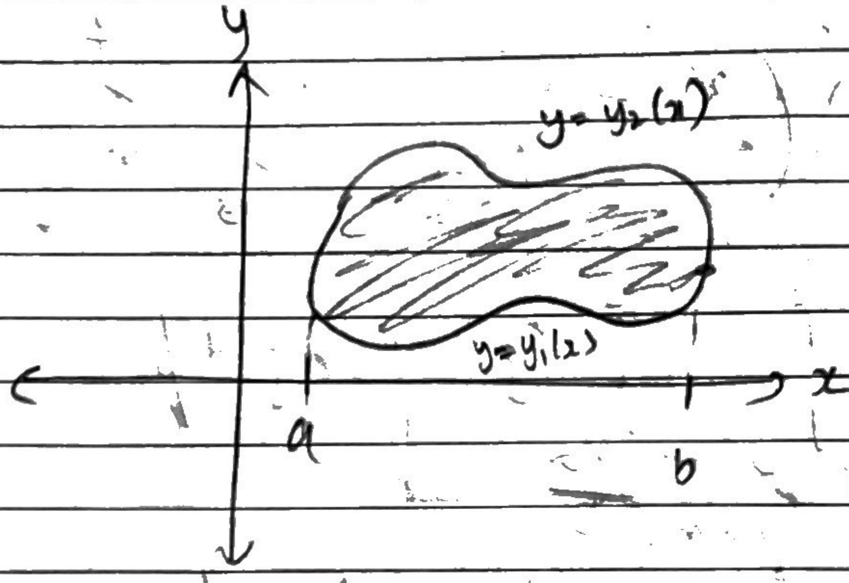
$$= 2\pi\left(\frac{3^5}{4} - \frac{3^5}{6}\right) = \pi\left(\frac{3^5}{2} - \frac{3^4}{3}\right)$$

Applications

1. Area

(i) The area of a plane region bounded by $y = y_1(x)$ and $y = y_2(x)$ between $x = a$ and $x = b$ is

$$A = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} dy\, dx,$$



2. Mass.

For a plane lamina, the surface density at the point $P(x, y)$ is $\rho = f(x, y)$. Therefore, the total mass of the lamina is given by $\iint \rho \, dx \, dy$.

In polar coordinates, taking $\rho = \phi(r, \theta)$ at the point $P(r, \theta)$. The total mass of the lamina is $\iint \rho\, r\, dr\, d\theta$.

store 67 — 24-7

3\. COM (plane) $\bar{X} = \dfrac{\iint x \rho \, dx \, dy}{\iint \rho \, dx \, dy}$ and $\bar{y} = \dfrac{\iint y \rho \, dy \, dx}{\iint \rho \, dx \, dy}$.

In polar form.

$$\bar{X} = \frac{\iint r\cos\theta \, \rho \, r \, dr \, d\theta}{\iint \rho \, r \, dr \, d\theta} \text{ and } \bar{y} = \frac{\iint r\sin\theta \, \rho \, r \, dr \, d\theta}{\iint \rho \, r \, dr \, d\theta}.$$

4\. <u>Moment of Inertia</u>

~~Particle~~ If a particle of mass m of a body is at a distance r from a given line, ~~it~~ then mr^2 is called the moment of inertia of the particle about the given line. ~~And~~ ~~$\Sigma m_i r_i$~~

If ~~an obj~~ a body of mass m is made of elementary masses m_i each at a distance r_i from the given line, then $\Sigma m_i r_i^2$ is the moment of inertia of the body about the given line.

Consider an elementary particle of mass $\rho_A \Delta x \Delta y$ at the point $P(x, y)$ of a plane area

MI about x axis = ~~B~~ $\rho \Delta x \Delta y \, y^2$

Total MI about x-axis,

$$I_x = \iint \rho y^2 \, dx \, dy$$

$$I_y = \iint \rho x^2 \, dx$$

$$I_0 = I_x + I_y = \iint \rho (x^2 \quad ^2) \, dx$$

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math N.B, chem H.W,
integration formulas
reduction form

5- Volume of a solid

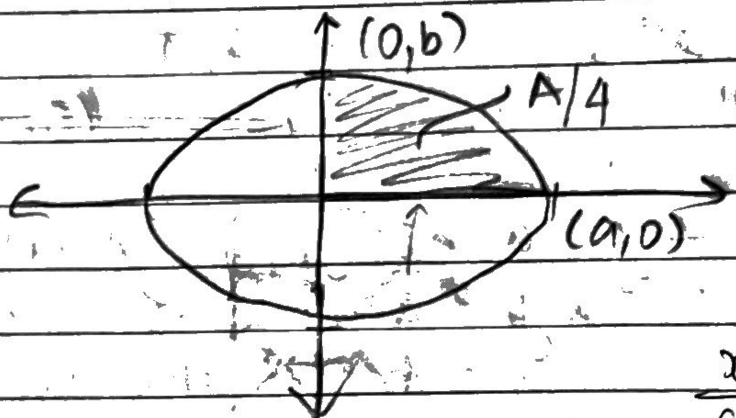
$$V = \iiint dx\,dy\,dz$$

$$V = \iiint r^2 \sin\theta \, dr\,d\theta\,d\phi$$

$$V = \iiint \rho \, d\rho\,d\phi\,dz$$

16.10.19

38. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a>b$ by double integration



$$\frac{x^2}{a^2} = \frac{b^2-y^2}{b^2}$$

$x: 0$ to $\frac{a}{b}\sqrt{b^2-y^2}$

$$x = \frac{a}{b}\sqrt{b^2-y^2}$$

$y: 0$ to b

$$A = 4\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$$

$$= 4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\,dy$$

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$$= \frac{4a}{b} \int_0^b \sqrt{b^2-y^2}\, dy$$

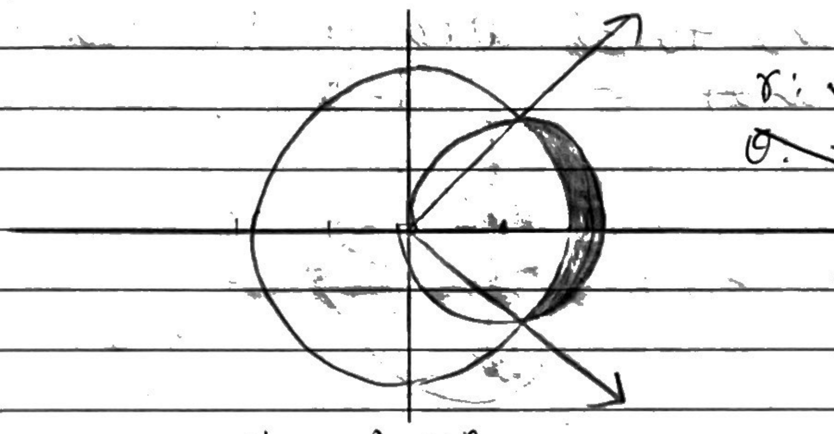
$$= \frac{4a}{b} \left[\frac{y}{2}\sqrt{b^2-a^2} + \frac{b^2}{2} \sin^{-1}\left(\frac{y}{b}\right) \right]_0^b$$

$$= \frac{4a}{b} \left(\cancel{\frac{b}{2} \times 0} + \frac{b^2}{2} \sin^{-1}(1) \right)$$

$$= \frac{4ab}{2}\left(\frac{\pi}{2}\right) = \boxed{ab\pi}$$

$$\boxed{A = \pi ab}$$

39. Find the area of the crescent bounded by the circles $r = \sqrt{3}$ and $r = 2\cos\theta$.



r: $\sqrt{3}$ to $2\cos\theta$

~~θ: $-\frac{\pi}{2}$ to $\frac{\pi}{2}$~~

Solving,

$$\sqrt{3} = 2\cos\theta$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \pi/6$$

$$A = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=\sqrt{3}}^{2\cos\theta} r\, dr\, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \left[\frac{r^2}{2}\right]_{\sqrt{3}}^{2\cos\theta} d\theta$$

$$= \cancel{2} \times \int_0^{\pi/6} \frac{4\cos^2\theta - 3}{\cancel{2}}\, d\theta = \int_0^{\pi/6} 4\cos^2\theta\, d\theta - \frac{\pi}{2}$$

$\cos 2\theta = 2\cos^2\theta$

$$A = 4\int_0^{\pi/6} \cos^2\theta \, d\theta - \frac{\pi}{2}$$

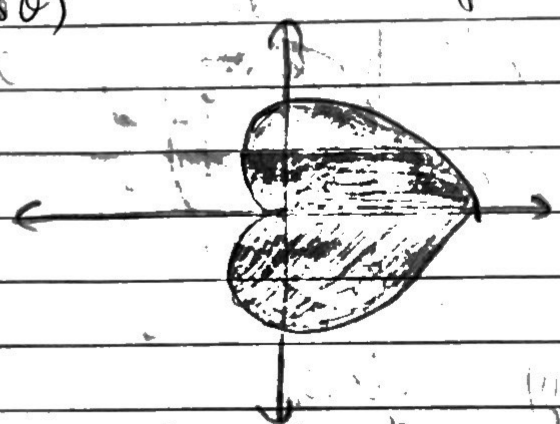
$$= \cancel{4}^{2}\int_0^{\pi/6} \frac{\cos 2\theta + 1}{\cancel{2}} \, d\theta - \frac{\pi}{2}$$

$$= 2\int_0^{\pi/6} \cos 2\theta \, d\theta + 2\int_0^{\pi/6} d\theta - \frac{\pi}{2}$$

$$= \frac{2}{2}\left(\sin\frac{\pi}{3}\right) + \frac{2\pi}{6} - \frac{3\pi}{6}$$

$$\boxed{A = \frac{\sqrt{3}}{2} - \frac{\pi}{6}}$$

40. Find the total area of the cardioid $r = a(1+\cos\theta)$



$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{a(1+\cos\theta)} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{a^2(1+\cos\theta)^2}{2} \, d\theta = a^2\int_0^{\pi} (1+\cos\theta)^2 \, d\theta$$

$$= a^2\int_0^{\pi} 4\cos^4 \theta/2 \, d\theta = 4a^2\int_0^{\pi} \cos^4 \theta/2 \, d\theta$$

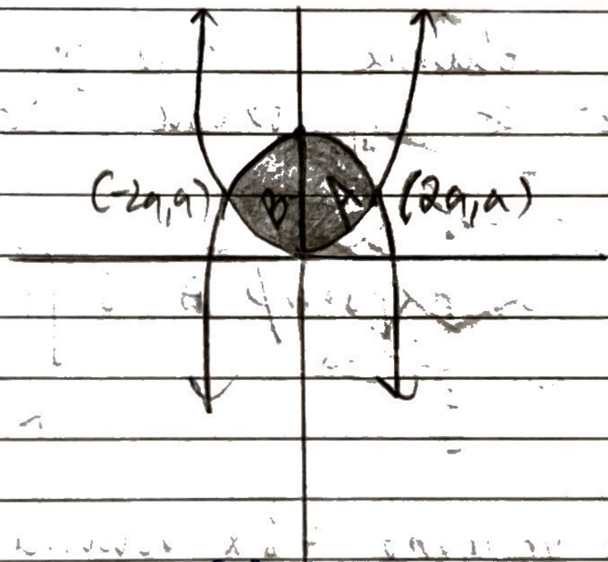
$$= \cancel{8a^2 \int_0^{\pi/2} \cos^4\frac{\theta}{2}\, d\theta} \qquad t = \theta/2 \quad dt = \frac{d\theta}{2}$$

$$= \cancel{16a^2 \int_0^{\pi/4}}$$

$$= 8a^2 \int_0^{\pi/2} \cos^4 t\, dt = \cancel{8}a^2 \left(\frac{3}{\cancel{4}}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{A = \frac{3\pi a^2}{2}}$$

41. Find the area bounded between the parabolas $x^2 = 4ay$ and $x^2 = -4a(y-2a)$



Solving:

$$\cancel{4a}y = -\cancel{4a}(y-2a)$$

$$y = -y + 2a$$

$$y = a$$

$$x^2 = 4a^2$$

$$x = \pm 2a$$

$$A = 2\int_{x=0}^{2a} \int_{y=\frac{x^2}{4a}}^{y=\frac{8a^2-x^2}{a}} dy\, dx$$

$$x^2 = -4ay + 8a^2$$

$$x^2 - 8a^2 = -4ay$$

$$y = \frac{8a^2 - x^2}{4a}$$

$$= \cancel{2}\int_0^{2a} \frac{8a^2 - x^2 - x^2}{2\cancel{4}a}\, dx = \int_0^{2a} \frac{8a^2 - 2x^2}{2a}\, dx$$

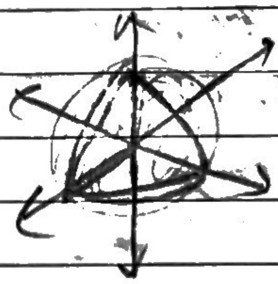
$$A = \int_0^{2a} \frac{4a^2 - x^2}{a}\, dx = \int_0^{2a} 4a\, dx - \frac{1}{a}\int_0^{2a} x^2\, dx$$

$$= 4a(2a) - \frac{1}{a}\left(\frac{x^3}{3}\right)_0^{2a}$$

$$= 8a^2 - \frac{8a^{\cancel{3}2}}{3\cancel{a}} = \frac{8a^2 \times 2}{3} = \frac{16a^2}{3}$$

$$\boxed{A = \frac{16a^2}{3}}$$

42. Find the volume of the sphere $x^2+y^2+z^2=a^2$ by triple integration.



$$A = 8 \iiint_R dx\,dy\,dz$$

Required volume $= 8 \times$ volume in the first octant

$$= 8 \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{8\pi}{2} \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta \int_0^a r^2\, dr$$

$$A = 4\pi \left[-\cos\theta\right]_0^{\pi/2} \left(\frac{a^3}{3}\right)$$

$$\boxed{A = \frac{4\pi a^3}{3}}$$

Using cartesian:

z : 0 to $\sqrt{a^2-x^2-y^2}$

y : 0 to $\sqrt{a^2-x^2}$

x : 0 to a

$$A = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= \frac{4\pi a^3}{3}$$

43. Find the volume of the cylinder $x^2+y^2=a^2$ and $0 \le z \le h$

Switching to cylindrical,

$$A = \int_0^h \int_0^{2\pi} \int_0^a \rho \; d\rho\, d\phi\, dz$$

$$= (h)(2\pi)\left(\frac{a^2}{2}\right) = \pi a^2 h$$

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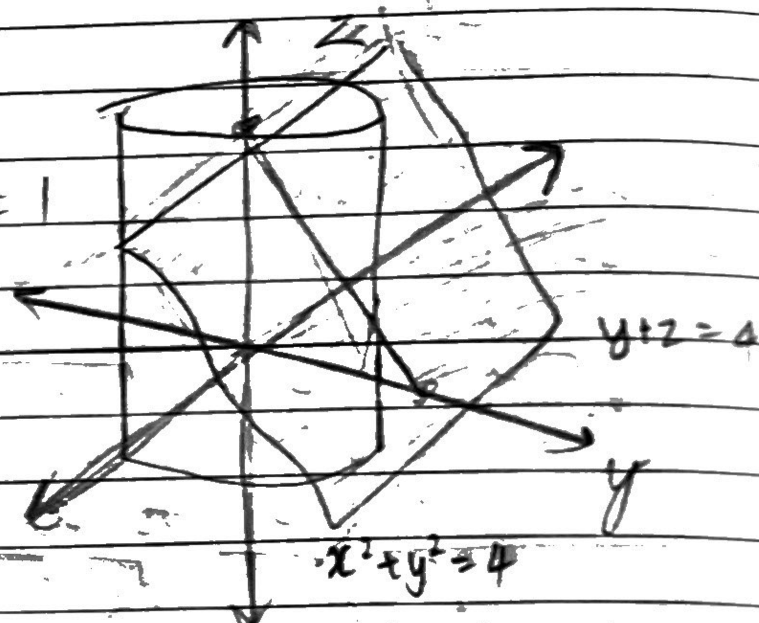
49. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=4$.

$\frac{y}{4}+\frac{z}{4}+0=1$



Switching to cylindrical

$\rho = 2 \rightarrow$ cylinder $\qquad z=z$

$\rho\sin\phi + z = 4 \qquad x=\rho\cos\phi$

$y=\rho\sin\phi$

$z = 4-\rho\sin\phi$

z: 0 to $4-\rho\sin\phi$

ρ: 0 to 2

ϕ: 0 to 2π

$$V = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{2}\int_{z=0}^{4-\rho\sin\phi} dz\,d\rho\,d\phi$$

$$= \int_0^{2\pi}\int_0^{2} 4-\rho\sin\phi \; d\rho\, d\phi$$

$$V = \int_0^{2\pi} \left[4\rho - \frac{\rho^2}{2} \sin\phi \right]_0^2 d\phi$$

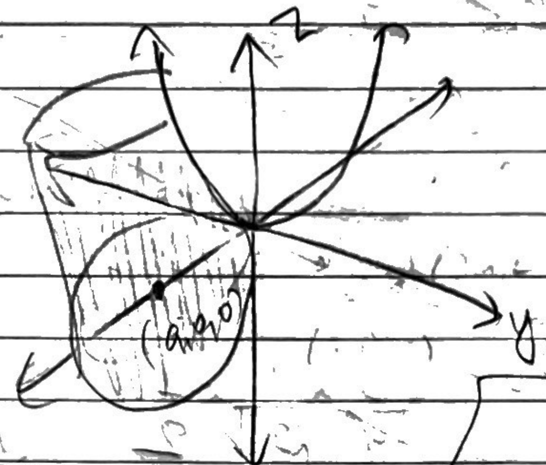
$$= \int_0^{2\pi} 8 - 2\sin\phi \, d\phi = 8 \times 2\pi + 2\left[\cos\phi\right]_0^{2\pi}$$

$$\boxed{V = 16\pi}$$

45- Find the volume of the cylinder $x^2+y^2=2ax$ intercepted between the paraboloid $x^2+y^2=2az$ and the x-y plane.

$x=z$

$x^2 - 2ax + a^2 + y^2 = a^2$



$(x-a)^2 + y^2 = a^2$

$z=0$ to $z = \dfrac{x^2+y^2}{2a}$

$$\int_{x=0}^{2a} \int_{y=-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \int_{z=0}^{\frac{x^2+y^2}{2a}} dz\,dy\,dx$$

$y = \sqrt{2ax-x^2}$

converting to cylindrical,

$x^2+y^2 = 2ax$

$\rho^2 = 2a\rho\cos\phi$

cylinder: $\boxed{\rho = 2a\cos\phi}$ $\phi : -\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\boxed{z = \frac{\rho^2}{2a}}$$

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$$I = \int_{-\pi/2}^{\pi/2} \int_{\rho=0}^{2a\cos\phi} \int_{z=0}^{\rho^2/2a} \rho \, dz \, d\rho \, d\phi$$

$$I = \int_{-\pi/2}^{\pi/2} \int_{0}^{2a\cos\phi} \frac{\rho^3}{2a} \, d\rho \, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{2a}} \times \frac{1}{\cancel{4}} \times 2 \cancel{a^4}^{\,3} \cancel{4} \cos^4\phi \, d\phi$$

$$= 2\int_{0}^{\pi/2} 2a^3 \cos^4\phi \, d\phi = 4a^3 \int_{0}^{\pi/2} \cos^4\phi \, d\phi$$

$$= \cancel{4}a^3 \times \frac{3}{\cancel{4}} \times \frac{1}{2} \times \frac{\pi}{2} = \boxed{\frac{3a^3\pi}{4}}$$

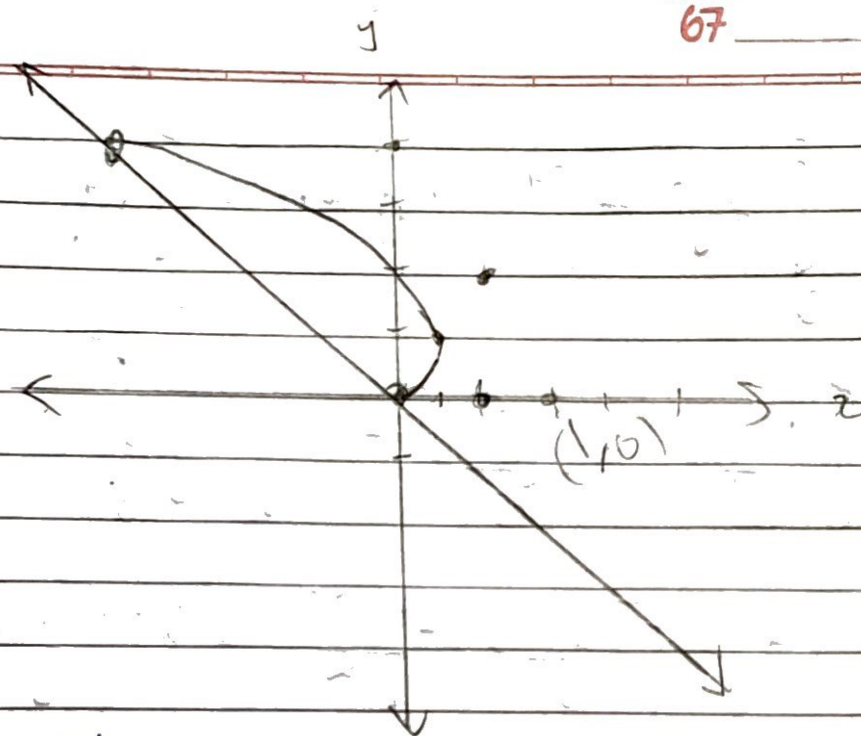
$$= \frac{3\pi}{8}$$

46. Find the mass and moments of inertia relative to x-axis, y-axis and the origin of the plane region having $\rho = x+y$ and bounded by the parabola $x = y - y^2$ and the line $x+y=0$

$$x = -(y^2 - y)$$

$$= -\left(y^2 - y + \frac{1}{4}\right) + \frac{1}{4}$$

$$x = -\left(y - \frac{1}{2}\right)^2 + \frac{1}{4}$$



$x = -y$

$-y = y - y^2$

$2y = y^2$

mass

$$M = \iint \rho \, dA = \int_{y=0}^{2} \int_{x=-y}^{y-y^2} \rho \, dx \, dy$$

$$M = \int_0^2 \int_{-y}^{y-y^2} (x+y) \, dx \, dy$$

$$= \int_0^2 \left[\frac{x^2}{2} + yx \right]_{-y}^{y-y^2} dy$$

$$= \int_0^2 \frac{(y-y^2)^2}{2} - \cancel{\frac{(y^2)}{2}} + y(y-y^2) + y^2 \, dy$$

$$= \int_0^2 (y-y^2)\left(\frac{y-y^2}{2}\right) + \frac{y^2}{2} + (y-y^2)(y) \, dy$$

$$= \int_0^2 \frac{y^2 + y^4 - 2y^3}{2} + \frac{y^2}{2} + y^2 - y^3 \, dy$$

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$$M = \int_0^2 2y^2 + \frac{y^4}{2} - 2y^3 \, dy = \int_0^2 2y^2 + \frac{y^4}{2} - 2y^3 \, dy$$

$$= \left[\frac{2y^3}{3} + \frac{y^5}{2 \times 5} - \frac{2y^4}{4} \right]_0^2$$

$$= \frac{2^4}{3} + \frac{2^4}{5} - \frac{2^{\cancel{5}4}}{2} = 2^4 \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2} \right)$$

$$= 2^4 \left(\frac{10 + 6 - 15}{30} \right) = \frac{2^4}{30} = \boxed{\frac{8}{15} = M}$$

$$I_x = \int_0^2 \int_{-y}^{y-y^2} (x+y) y^2 \, dx \, dy$$

$$= \int_0^2 \int_{-y}^{y-y^2} xy^2 + y^3 \, dx \, dy$$

$$= \int_0^2 \left[\frac{x^2 y^2}{2} + y^3 x \right]_{-y}^{y-y^2} dy$$

$$= \int_0^2 \frac{(y-y^2)^2 y^2}{2} + (y-y^2) y^3 - \frac{y^4}{2} + y^4 \, dy$$

store
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$$= \int_0^2 \frac{(y^2+y^4-2y^3)y^2}{2} + y^4 - y^5 - \frac{y^4}{2} + y^4 \, dy$$

$$= \int_0^2 \frac{\cancel{y^4}+y^6-2y^5}{2} + y^4 - y^5 - \frac{\cancel{y^4}}{2} + y^4 \, dy$$

$$= \int_0^2 2y^4 - 2y^5 + \frac{y^6}{2} \, dy = 0.6095$$

$\cancel{-672}$

$$= \left[\frac{2y^5}{5} - \frac{2y^6}{6} + \frac{y^7}{14}\right]_0^2$$

$$-\frac{32}{315}$$

$$= \frac{2^6}{5} - \frac{2^6}{3} + \frac{2^6}{7} = 2^6\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right)$$

$$= 2^6\left(\frac{21-35+15}{7\times3\times5}\right) = \frac{2^6}{7\times15} = \boxed{\frac{64}{105}}$$

$$I_y = \int_0^2 \int_{-y}^{y-y^2} (x+y)x^2 \, dx \, dy = \int_0^2 \int_{-y}^{y-y^2} x^3 + yx^2 \, dx \, dy$$

$$32\left(\frac{1}{15} - \frac{1}{315}\right)$$

$$I_y = \int_0^2 \left[\frac{x^4}{4} + \frac{yx^3}{3}\right]_{-y}^{y-y^2} dy = \frac{32\times 300^{20}}{\cancel{15}\times 315}$$

$$= \int_0^2 \frac{(y-y^2)^4}{4} + \frac{y(y-y^2)^3}{3} - \frac{y^4}{4} + \frac{y^4}{3} \, dy$$

$$= \frac{+96}{3\times35} - \frac{64}{63} + \frac{32}{15} - \int_0^2 \frac{y}{3}\left(y^3 - 3y^4 + 3y^5 - y^6\right)$$

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$$I_y = \int_0^2 \frac{(y-y^2)^4}{4} + \frac{(y-y^2)^3}{3} - \frac{y^4}{4} + \frac{y^4}{3}\,dy$$

$$= \frac{184}{315} + \frac{-52}{105} - \frac{2^5}{5\times 2^2} + \frac{2^5}{5\times 13}$$

$$= \frac{184}{315} - \frac{52}{105} - \frac{8}{5} + \frac{32}{65}$$

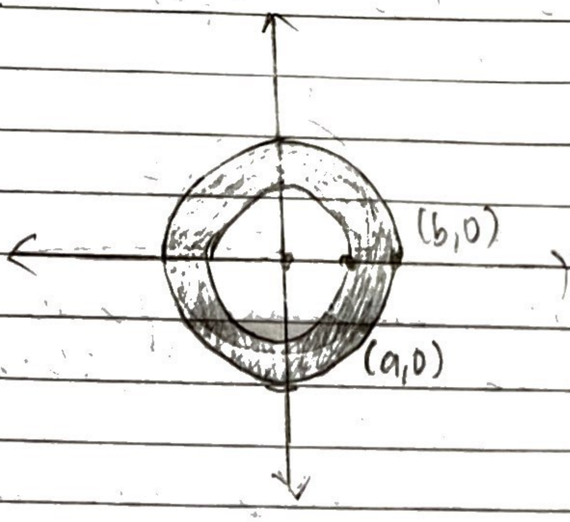
$$= \frac{4}{45} - \frac{72}{65} = \frac{-596}{585}$$

$$I_y = \int_0^2 \frac{(y-y^2)^4 - y^4}{4} + \frac{y(y-y^2)^3}{3} + \frac{y^4}{3}\,dy$$

$$= \int_0^2 \frac{((y-y^2)^2+y^2)((y-y^2)^2-y^2)}{4} + \frac{y}{3}\left((y-y^2)^3+y^3\right)dy$$

$$= \int_0^2 \frac{(2y^2+y^4-2y^3)(2y-y^2)(-y^2)}{4} + \frac{y}{3}($$

47. Prove that the moment of inertia about an axis through the centre perpendicular to the plane of a circular ring whose inner and outer radii are a and b is $\frac{M}{2}(a^2+b^2)$ where M is mass of the ring.



$$\rho = \frac{\text{mass}}{\text{area}} = \frac{M}{\pi(b^2-a^2)}$$

$$\rho = \frac{M}{\pi(b^2-a^2)}$$

$$I_0 = \int \int \rho (x^2+y^2)\, dx\, dy$$

switching to polar

$$I_0 = \int_{\theta=0}^{\theta=2\pi} \int_{r=a}^{r=b} \rho\, r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_a^b r^3 dr \left(\frac{M}{\pi(b^2-a^2)}\right)$$

$$= (2\cancel{\pi})\left(\frac{M}{\cancel{\pi}(b^2-a^2)}\right)\left(\frac{b^4-a^4}{4}\right)$$

$$\boxed{I_0 = \frac{M(a^2+b^2)}{2}}$$